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The Mereotopological Structure of the Brentanoraum B^3 and Material Entities

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Abstract

The pure space is filled with extended three-dimensional material entities like a car, a chair or the Eiffel Tower which occupy spatial regions. The perception of these objects is linked to the perception of their two-dimensional material boundaries which demarcate them from their surroundings. The purpose of my diploma thesis is the development of an axiomatic theory of the surrounding space and its embedded material entities which is adequate to our cognition.

The universe of discourse of the Brentanoraum $\mathbf{B}^3$ (our theory of space) is divided into four classes, namely three-dimensional space regions, two-dimensional surface regions, one-dimensional line regions and zero-dimensional point regions. This theory is based on first order logic with identity enriched by the primitives: *spatial part*, *spatial coincidence*, *space region* and *spatial boundary*. With the help of this framework we will give a detailed classification of spatial entities with respect to their mereological, topological and morphological properties. A ancillary result is that we find a way to distinguish the occupied space regions of a scoop of ice cream and a doughnut without adding a new basic relation like "genus" or "handle".

Since ancient times the essence of material entities has been discussed. On the basis of an extensive literature review we will analyse material entities with respect to their ontological status. Our point of view is that a certain material structure is an individual that fulfils the following conditions: it is a presential, it is a bearer of qualities, it occupies space and it consists of a presential amount of substrate. On the basis of these assumptions we will extend the Brentanoraum to a spatial-material theory. The main focus of this theory is on material boundaries. We assume that they are cognitive items which do not belong to the physical level of reality. That means material boundaries are not three-dimensional material parts of a material structure but rather lower-dimensional entities without an amount of substrate.

Altogether, this thesis provides an axiomatic foundation of an ontology of spatial and material entities.

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Chapter 1

Introduction

1.1 General Remarks

Ontology is the study of two questions: 1. What exists? and 2. What is the mode of its being? The main task is the development of a hierarchy of basic types of entities, so-called categories and the clarification of their structural relations.

Formal Ontology[1] is an aggregation of methods of mathematical logic and classical ontology. The aim is the representation of the ontological structure of the world or facets of reality by an axiomatic theory, expressed in a formal language like first-order, second-order or description logic. The results of such representations are also called ontologies and they can be differentiated by their level of abstraction. Very general theories are called *Top Level Ontologies* and they deal with the most basic categories of reality like space, time, quality or process. Hence, these general theories build a semantical and logical framework for ontologies with a more specified domain like a Biological Species Ontology. The utilization of ontologies in fields like biology, economy and medicine is common practice.

The General Formal Ontology (GFO) [Her, Hel, Bur, Hoehn, Loe, Mich 2006] is a Top Level Ontology which is developed by the Onto-med Research Group[2] in Leipzig. This work is a contribution to GFO but note that the results of this work have an independent and general character.

[1]Note that the term "Formal Ontology" is here used in a different sense than in philosophy. Our notion of the term is equal to its use in artificial intelligence and knowledge representation.

[2]Research Group around Prof. Dr. Heinrich Herre. For more details see www.onto-med.de.

1.2 Structure of the Thesis

In the second chapter we will summarize and analyse two famous ontological problems, namely the "problem of universals" and the "problem of objecthood". Furthermore we will present our philosophical assumptions and give a short overview about different categories used in GFO.

The third chapter exemplifies the cognitively inadequateness of the real space $\mathbf{R}^3$ as a model of the surrounding space. We will introduce the ideas of Brentano which offer an alternative description of the pure space. Furthermore we will talk about mereological and mereotopological systems which represent a framework for our theory of space.

Chapter four is one central part of this thesis. We will present our theory of spatial entities, the so-called "Brentanoraum $\mathbf{B}^3$". A number of definitions, axioms and theorems are presented. In subsection 4.6 we will give a first classification of spatial entities by using the mereotopological elementary equivalence relation.

The second central part is presented in chapter five. Here we will extend our spatial theory to a spatial-material theory. The first step is a detailed analysis of the essence of material structures and their belonging material boundaries as well as their interrelations to spatial entities. After an elaborate talk about granularity and the granularity function we will again present a number of definitions, axioms and theorems.

The last chapter summarizes the main results of this thesis and compares our approach with [Smi, Var 2000]. Ideas for future research are presented, too.

Chapter 2

Ontological Views

What actually exists and what are the modes of being? These are the central questions of general metaphysics or ontology. The term "ontology" was introduced by Rudolf Göckel[1] in [Goc 1613] but the first investigation and publication of beings can be traced back to Parmenides[2] in his tripartite poem[3]. Two of the most fundamental and oldest problems of ontology, namely "problem of universals" and "problem of objecthood" will be discussed in more detail in this chapter. Both problems are solved only insufficiently until today.

2.1 Problem of Universals

Mick Jagger, Keith Richards, Charlie Watts and Ron Wood are the members of the famous band "The Rolling Stones"[4]. What they have in common? Except from their genius of music the most obvious thing is that they are human beings. To come back to ontological views we can ask: What kind of interrelation between the general term "human being" and the individuals Mick, Keith, Charlie and Ron which are human beings exists? Is the general term "human being" only a linguistic expression or is it a so-called universal with an ontological existence? Philosophers analyzed this topic since ancient times and in fact there are three important tendencies: realism, nominalism and conceptualism.

What is a universal? For a better understanding of universals we contrast them with indi-

[1]Rudolf Goclenius the Older (1547-1628) was a german philosopher and professor of logic at the university of Marburg.

[2]Parmenides of Elea (early 5th century BC) was a pre-socratic greek philosopher.

[3]Unfortunately it has survived only in fragments (See [Mans 1985]).

[4]English rock band which was formed in 1962. They have sold more than 200 million albums worldwide.

viduals. Universals are abstract and mind-independent entities like qualities or relations, which can be exemplified by individuals. With the help of them we can explain relations of qualitative identity and resemblance among individuals. Individuals in contrast are singular objects which are not multiexemplifiable and non-repeatable respectively. This means that they cannot be in two places at the same time, unlike universals which can be exemplified by different individuals at the same time.

2.1.1 Realism

All versions of realism claim that yes, there are universals. They are existent or real and distinct from the individuals that instantiate (exemplify) them. The question of dependent or independent existence of the universals divides realism into two major forms.

Extreme Realism

Extreme realists claim that universals have an independent existence. The most famous and oldest version is the Platonic realism.

Plato‘s[5] two-world-theory claims that there is the world of universals (”mundus intelligibilis”), which Plato would call *Forms* or ideas, and the world of sensible objects (”mundus sensibilis”). Forms are immaterial, changeless and not mental entities. They are outside of space and time. A Form is a ”one-over-many”, that means to every collection of things to which we give the same name, like human beings or large things, there is a unique single Form. The things we perceive in the world are only shadows of the real things, they participate in Forms. That means Forms can be understood as archetypes and a specific human being is only a copy of them. For Plato universals are preexistent and that is why his approach is a theory of *universalia ante rem*.

The most powerful argument against the Platonic realism is the *Third Man Argument* which comes from Plato himself. According to Plato‘s belief we say that the members of the Rolling Stones are human beings because they participate in the single Form F1 which we call human being. Consider now the collection of Mick Jagger, Keith Richards, Charlie Watts, Ron Wood and the single Form F1. How can we explain that every member of this collection can be called a human being? The consistent answer has to be that we need another single Form F2 and this Form will be the ”third man”. The existence of the

[5]Plato was a greek philosopher (428/427-348/347 BC) and a student of Sokrates

second Form F2 contradicts the assumption of uniqueness and furthermore the repetition of the argument leads to an infinite regress.

Strong Realism

Aristotle[6] try to overcome the Third Man Argument by rejecting the assumption of the independent existence of Forms. The Aristotelian realism is the grounding of every strong realism belief.

Aristotle denies the preexistence of universals. He believes that a universal is just the quality that is in the individual and any other qualitatively identical individual. It is not independent from the individuals that have this quality, that means in contrast to Plato his universals are in space and time. Roughly speaking the universal and the quality coincide in the individual and they can only be separated by abstraction (by an intellectual act). That is why his approach is a theory of *universalia in rebus*. Note that an universal can exist in many places simultaneously and it is wholly present in each place.

The price for the immunity against the Third Man Argument is the assumption that universals are able to be in many places at once. Another objection to realism in general is the argument of the ontological sparingness that means a theory without universals, which also can explain qualitative identity and resemblance among individuals, is "better" than a theory with universals.

2.1.2 Nominalism

Nominalism is the counterpart of realism. Supporter of nominalism deny the real existence of universals and claim that only individuals exist. In fact, there are three different strategies to explain relations of qualitative identity and resemblance among individuals without universals, namely Predicate, Resemblance and Trope Nominalism. Nominalism is the main contribution to the problem of universals by medieval philosophers.

Predicate Nominalism

Mick Jagger and Keith Richards are human beings because the predicate "is a human being" can be truly said of both, but they do not have any entity in common. Predicates

[6] He was a greek philosopher (384322 BC) and a student of Plato.

or universals are only words for talking about individuals, they are just names. Asked why Mick and Keith are both called human beings and not animals, a predicate nominalist would say it is just a non-explainable basal fact.

Roscelin[7] is known as the originator of Predicate Nominalism which is also called Ostrich or Extreme Nominalism. His nominalistic view lead him to his doctrine on the trinity, namely that "trinity" is just a word and God the Father, God the Son and God the Holy Spirit are separate individuals and no one being. The church was not amused about that and that is why he was ordered by the Synod of Soissons to recant in 1092.

Resemblance Nominalism

Qualities are classes of resembling individuals. Qualitative identity and resemblance among individuals are explained by belonging to the same class. Mick Jagger and Keith Richards are human beings because both are members of the class (set) human beings. Note that classes are not universals because they are not repeatable.

One problem of Resemblance Nominalism is caused by the extensional principle of set theory. Consider the sets constructed for the property "being a member of the Rolling Stones" and for the property "playing the music for the film 'Shine a light'[8]". Both sets have exactly the same members and that is why a Resemblance Nominalist has to say "being a member of the Rolling Stones" and "playing the music for the film 'Shine a light'" is the same property.

Trope Nominalism

Trope Nominalism claims that there are also particular properties beside particular things. These properties ,so-called "tropes", cannot be shared between individuals because they are only in one place at a time. Mick Jagger and Keith Richards are human beings because both have a "being a human being" trope "in" them, which are numerically distinct but qualitively identical. A particular is seen as a complex of tropes.

Critics object that the explanation of qualitative resemblance among individuals by qualitative identical tropes "in" the individuals is only a shifting of the problem and no expla-

[7]Johannes Roscelinus of Compiègne (ca. 1050-1124) was a french philosopher and theologian.
[8]A documentary film about the Rolling Stones (2008).

nation at all.

2.1.3 Conceptualism

The third "solution" of the problem of universals is the conceptualism, it is a position between the abstract realism and the ontological sparingly nominalism. General terms or universals are neither reality (Platonic Realism) nor just words (Nominalism), but rather concepts in mind. Concepts are constructed by abstraction of similar individuals. Mick Jagger and Keith Richards are human beings because both are instances of the mental object "human being", in other words the concept "human being" apply to Mick and Keith.

Critical remarks about conceptualism deal with questions like: How can we explain that a concept applies right? What about generality of concepts?

2.2 Problem of Objecthood

Objecthood is the state of being an object, therefore the meaning of "objecthood" depends on or can derive from the meaning of "object". In fact, there are two questions to ask: Is the existence of the object independent of their properties? And second in case of yes: What is the nature of this existence? The main approaches are *Bundle Theory* and *Substance Theory*, others are the *Neoaristotelian Substance Theory* and the *Theory of Individual Essences.*

2.2.1 Bundle Theory

An object is nothing more than a collection or bundle of qualities and relations to other objects. That means an object is nothing beside the properties, it cannot be understood as something separated and an "object" without properties is no object. On the other side an individual property either can‘t exist separatly from such a bundle.

Mick Jagger is only a bundle of certain qualities such as his individual hairstyle, body height, color of the skin and voice volume but if he cut his hair a bundle theorist has to say that he would not be the original object anymore, he would be another Mick Jagger.

2.2.2 Substance Theory

Substance Theory suggests that something underlies the properties and relations - something distinct of them a so-called *substance*. The substance is connected to the properties and relations by the inherence relation. What is the notion of this inherence relation? The expression "quality x inheres in substance y" can be understood as that the substance y *has* the quality x in the sense of the substance is in property of the quality. The substance y is the bearer of this quality x but the quality is no part of the substance. That is why it is also called a "bare particular".

The strongest objection is dealing with the "Propertylessness" of the substance. How can we conceive a propertyless thing? Every time when we have something in mind, we think of some property. It is just impossible to think about a propertyless thing.

2.3 Our Point of View

In this section we want to give an overview about our philosophical position. According to the aim of this thesis we will discuss material entities in more detail.

2.3.1 Philosophical Assumptions

The main assumption is the independent existence of the external world. The external world does not need a perceiving subject for its existence, it is independent of our minds. This approach is called *Ontological Realism*.

Assuming a naive point of view, material entities like a car, a chair or a cigarette belong to the external world. To clarify the famous subject-object relation and the relation between the external and the perceived world respectively we have to answer the following questions at first, namely: 1. What is it what we call a material entity (compare section 2.2)? and 2. What is meant by the term (perceiving) subject?

We want to use the term subject for every entity with perceptive abilities. In this sense a mosquito is a subject as well as a human being. An object in general is something which can be perceived by an subject. Note that an ontological subject also can be an object, it depends on context. If we talk about material entities we want to distinct between the "thing-in-itself" like Immanuel Kant[9] would say and the phenomenal material entity. The thing-in-itself is the material entity *per se*, it is independent of the observer. We call it "urobject". On the other side we have the phenomenal object that can be understood as a set of unfold dispositions of the urobject. A disposition unfold in the thing-in-itself if and only if there is a perceiving subject and we will call this unfold disposition an attribute.

Consider a house, a human being "Frances" and a mosquito "Wasi". Frances and Wasi have different perceptive abilities, e.g. the different observable wavelength range of a human and a compound eye. Imagine now that both are "looking" to the house, Frances and Wasi do not see the same house, there are two distinct phenomenal objects. The perceived houses of Frances and Wasi differ for instance in the unfold disposition "form". Note that these phenomenal houses are only two of the infinitely possibilities of phenomenal objects of the urobject house. It depends on the observers perceptive abilities (and auxiliary material like an electron microscope) which phenomenal object comes to his minds. We can summarize this in a triple-digit relation Rel(O_P,O_U,S_P), whereas O_P is the phenomenal object , O_U the urobject and S_P the perceiving subject. The phenomenal object connects the perceiving subject with the urobject.

To know all forms of appearance of an urobject is the same as to know the absolute truth about an urobject. That is just impossible because the perceptive forms of appearance are only a small part of all forms of appearance.

2.3.2 Universals in GFO

GFO has a general distinction between individuals and universals. The class of universals and the class of individuals are disjoint. An individual is a thing in space and time like a material entity and a universal is an entity that can be instantiated by different individuals. There are three kinds of abstract entities (universals) in GFO, namely *immanent universals*, *conceptual structures* and *symbolic structures*, which are connected by different ontological relations. This classification represents a pluralistic approach, that means we are not forced ourselves to realism, nominalism or conceptualism.

Immanent universals are constituents or invariants of the objective world. Their instances are classes of "similar" individuals that means there is a property or a set of properties

[9]Immanuel Kant (1724-1804) was a german philosopher.

which characterize them. These properties can only be perceived by subjects, but note that they are independent of them in the sense that properties have a material foundation, which has an independent existence of the perceiving subjects.

If we observe an individual like a specific house, then we perceive a certain phenomenal house with a certain color and shape. These properties are subjective properties and not independent of the objective properties of the individual. They are the foundation for the construction of a concept (conceptual structure). Note that we distinct between the abstract concept and the individual representation of this concept. An abstract concept arises because of communication among subjectives that means language and communication are essential for the developing process of concepts. An universal is understood via an individual representation of an concept. This relation is called correlation.

A symbol or a symbolic structure is independent of space and time, it is abstract. Their instances are called *token*. The letter "A" has two different significances at the same time: 1. It is a token of the abstract symbol "A". and 2. It is a representation of the abstract symbol "A". There is a relation between the symbolic structure(house) and the material entity(house) which is called denotation. This relation can only be activated by instantiation of the abstract symbolic structure "house" by articulation or transcription.

In consideration of the assumptions above we have to distinct between the material entity(house), the phenomenal object(house), the immanent universal(house), the conceptual structure(house), the individual representations of the concept(house), the symbolic structure(house) and the tokens(house). A deeper investigation of the ontological relations like instantiation, correlation, representation and denotation is given in [Her, Knu, Loe 2008].

Chapter 3

The Structure of Space

3.1 Preliminary

The pure space is filled with extended three-dimensional material entities like a car, a chair or the Eiffel Tower which occupy spatial regions. The perception of these objects is linked to the perception of their two-dimensional boundaries (material surfaces) which demarcated them from their surroundings. Boundaries are one main constituent of the common-sense picture of the world and a theory of space has to answer several problems, e.g.: How to describe the contact between two objects? Is a boundary of an object also a boundary of the adjacent object (symmetry)? Further considerations deal with questions of granularity and vagueness.

3.2 The Standard Model - The Real Space $\mathbf{R}^3$

The standard model of the surrounding space is the real space $\mathbf{R}^3$. That means spatial regions which are occupied by objects are subsets of the real space. We will show that this approach is contradictorily and cognitively inadequate to describe the structure of space.

If we talk about the occupied space regions (subsets of $\mathbf{R}^3$) we have to claim at least the following assumption: The intersection of space regions which are occupied by two "different" objects is empty (A). Different objects means that we exclude the situation that one object is a material part of the other. It is adequate to cognition that the occupied space regions of such objects are disjoint.

3.2.1 Open/Closed Distinction

In the real space we distinguish between open and closed subsets. The *unit ball* D^3 is a subset of the real space which contains all points of the $\mathbf{R}^3$ with an distance less or equal to one. It is an example for a closed subset. The subset which result by excluding points with an distance equal to one is a open subset. Note that there are also subsets which are neither closed nor open.

If we talk about space regions which are occupied by objects we have to decide between an open or closed occupied subset. One may say that the distinction between open and closed objects does not make sense and we postulate that all objects are closed or all objects are open (or neither/nor). However to make the decision we must exclude arbitrariness. Therefore the following weak assumptions: The occupied space regions of "one sort" of objects are either open or closed (B). The assumption B excluded the possibility that the occupied space region of one Volvo 240 is open and the occupied space region of another Volvo 240 is closed, that would mean a peculiar privileging.

3.2.2 Contact in the Standard Model

How can we model the notion of contact in the $\mathbf{R}^3$? A minimum requirement of contact between two different objects is the following: The space regions which are occupied by two different objects which are in contact following one another immediately (C). This means that there is no gap or cavity between their occupied space regions in the relevant contact area. It is hardly imaginable that two objects in contact do not fill out the whole space in the relevant contact area, it is just a basal fact.

Consider two different cubes x and y which lie upon each other. This means that both cubes are in contact. The lower cube x is colored in red and the upper cube y is colored in white. The embedding of both cubes in the real space leads to four interpretations with respect to the open/closed distinction (in the relevant contact area).

1. The white cube is closed and the red cube is open.
2. The white cube is open and the red cube is closed.
3. Both cubes are open.
4. Both cubes are closed.

The first and second interpretation are prohibited by assumption B (arbitrariness). There is no argumentation which legitimates the choice that the white cube is closed and not the red one or vice versa. The third and fourth interpretation conflicts with the assumption A (empty intersection) and C (following one another immediately). If the closed or open occupied space regions have an empty intersection then via using the density of the continuum we can construct a nonempty subset between them. If they follow one another immediately then we cannot have an empty intersection by using that both are either open or closed.

We have shown (under weak reasonable assumptions A, B, C) that the embedding of objects in the real space $\mathbf{R}^3$ cannot explain contact between two objects. In the next subsection we will include boundaries in our consideration.

3.2.3 Boundaries in the Standard Model

The perception of objects is linked to the perception of their boundaries which mark them off their surroundings. The standard topological definition of a boundary of a subset S of the real space is a set of points so that every open neighborhood of these points intersects S and the complement of S. That means boundaries (in topological sense) are symmetric.

Consider the cube-example in the subsection above. By using the topological definition of a boundary results that the boundary of the white cube is the same as the boundary of the red cube. This interpretation is problematic because we can ask: What color has this boundary? If it is white, than we have the strange issue that a red cube has a white boundary and vice versa. One may say that the color is "whed" (white and red at the same time) or magenta (combinated color of white and red). The color magenta leads again to the strange conclusion that the red cube has a magenta boundary. Beside the difficulty to imagine a color like "whed" we will do the following thought experiment: Exchange the white cube for a black one. That means the boundary of the red cube is now "bled" (black and red at the same time) but we did not modify the red cube. The last loophole is to say that boundaries do not have qualities like a color. But it is just a basal fact that we observe a red boundary (surface), if we look to a red cube.

To get out of the problem of symmetry one may say that only inner boundaries are boundaries of objects. This means that only closed objects have a boundary. Again we are in need of justification why some objects are open and some are closed which means that some objects have a boundary and others do not. But even if we assume that at least all everyday objects like a chair or a settee have an own boundary we have to postulate that all these objects are closed. By using the results of subsection 3.2.2 it is impossible to model contact between these ordinary objects.

3.2.4 Advanced Objections

The notion of the surrounding space is a notion of a *continuum*. That means between two objects which are not in contact exists an interjacent space region. The structure of space is continuous. The standard model $\mathbf{R}^3$ simulates space regions as sets of points, whereas points are so-called *urelements*. This is a down-to-top approach.

The following three arguments put a question mark over the ontological adequateness of a space-simulation based on set theory.

1. Points in the real space are unexpanded elements. But how can we construct an expanded space region out of unexpanded points?

2. The urelements in the space which surround us are extended three-dimensional objects and not unexpanded points.

3. A set theoretical approach cannot distinguish between an object and their boundary because both are interpreted as sets of points.

Note that these objections do not deny the usefulness of a space-simulation by the real space. In natural sciences it is common practice to measure distance or extension with the real numbers.

3.3 The Brentanian Approach

Franz Brentano (1838-1917) was a famous philosopher and psychologist. He taught at the university of Vienna. Edmund Husserl the founder of phenomenology was among his students. Our approach of space is inspired by the ideas of Brentano [Bren 1976] and Chisholm [Chis 1984].

3.3.1 Boundaries and Coincidence

Brentano rejects the association of the surrounding space as a mathematical continuum and therewith the open/closed distinction. All extended bodies have their own boundary, that means they can be treated as "closed" objects. He represents a top-to-down approach. The perception of the surrounding space is a perception of primary entities (extended continuous three-dimensional objects) and their belonging two-, one- and zero-dimensional boundaries (surfaces, lines and points). Note that Brentano uses the term *boundary* also for points on the surface area of a ball (outer boundary) and even for points inside the ball (inner boundary). In our considerations we will call these kinds of entities *hyper parts*. Surfaces and lines themselves are continuous and have lower-dimensional boundaries. Points are the only kind of boundaries which are unextended and therefore not continuous. Consider the following citation:

> "...eine Grenze, auch wenn sie ein Kontinuierliches ist, doch nie ohne Zugehörigkeit zu etwas Kontinuierlichem von mehr Dimensionen bestehen kann, ja durch die Weise dieser Zugehörigkeit erst ihren völlig bestimmten und genau spezifizierten Charackter empfängt..."(cited in [Bren 1976] p.16)

That means boundaries cannot exist in isolation, they are dependent entities. This is what we call the **1. Brentanian Thesis**.

An important property of boundaries is the possibility of coincidence. Two different boundaries are coincident if they are co-located. The cube-example in subsection 3.2.3 can be solved in an elegant way. The red and white cube have their own two-dimensional boundaries and their white and red surface are coincident. That means if two objects x and y are in contact, then there is a boundary of x and a boundary of y which are coincident. This is what we call the **2. Brentanian Thesis**. Note that Brentano himself did not distinct explicitly between material and spatial boundaries. In our consideration coincidence is only a property of spatial entities.

3.3.2 Plerose and Teleiose

Plerose and *Teleiose* are fundamental terms in Brentanos considerations about space and time. The first one is closely connected with the notion of coincidence of boundaries and the second one is dealing with differences of variation of continuas.

A boundary is always a boundary in a certain *direction*, namely in the direction of their belonging higher-dimensional entity. The Plerose of a boundary is a degree for the *amplitude of a boundary.* Consider the following citation:

> "Eine Grenze, die eine Grenze nach allen Richtungen ist, für die sie überhaupt eine Grenze sein kann, existiert in 'voller' Plerose; andernfalls existiert sie nur in mehr oder weniger 'partialer' Plerose"(cited in [Bren 1976] XXI Introduction)

That means Plerose can be defined as the proportion of number of interpretable directions to all possible directions. Consider the cube-example in subsection 3.2.3. The red and white boundary (surfaces) are boundaries with half-Plerose because both are only boundaries in one of two possible directions. A red surface which divides the red cube into two parts is a boundary with full-Plerose.

The fundamental term Teleiose can be interpreted as a degree of *perfection of a certain property.* Consider a parked car, a slow-moving car and a car driving on a highway. Brentano would say that the parked car is located at a certain place in complete Teleiose. The slow-moving and the fast-moving car are in incomplete Teleiose at a certain place and moreover the degree of Teleiose of the fast-moving car is lower than the slow-moving car. Note that we had to include the time-dimension for this consideration. Therefore the following citation:

> "Das Räumliche zeigt Unterschiede des Variationsgrades, hat eine variable Teleiose (Geschwindigkeit des Wechsels) nur in Rücksicht auf seine 4. Dimension, als Grenze derselben,..."(cited in [Bren 1976] p.32)

3.4 Mereology

Mereology is the theory of parthood relations or the theory of parts and wholes. The investigation of principles of an entity and his parts is the main task. What are the basic principles of the parthood relation? Consider the following statements:

1. The Iraq War was (is still) part of the policy of Georg W. Bush.

2. Germany is a part of Europe.

3. The bell is a part of a bicycle.

4. The natural numbers are part of the real numbers.

5. The second half is a part of a soccer match.

These examples show that the part-of-relation is used for different issues and in an ambiguous way, i.e. political, spatial, material, set theoretical and temporal part-of. Therefore the part-of-relation cannot be uniquely determined and one has to take the special domain/situation into consideration.

The first informal treatment of mereology can be traced back to Plato's dialog *Parmenides*[1]. The origin of formal mereology are the works of Lesniewski [Les 1916] and Leonard/Goodman [Leo, Good 1940] in the first half of the 20th century. Lesniewski provided an alternative set theory[2] which avoids antinomies like the famous Russel antinomy[3] without narrowing the notion of Cantors term "set". Up to the paper of Leonard and Goodman, his works were almost unstudied because he only published in Polish.

A mereological system is a theory based on first order logic whose universe of discourse consists of wholes and their respective parts of an arbitrary domain, enriched by a binary predicate, which represents the part-of-relation. Note that wholes and parts are not different kinds of entities like a set and a point in classical set theory. The *ground mereology* is a mereological system which satisfies at least the axioms of an partial ordering, that means reflexivity, symmetry and transitivity is claimed for the part-of-relation. In this sense the real numbers with the usual less-equal ($\mathbf{R}$,$\leq$) is an example for a minimal mereological system. Additional axioms and therefore extensions of ground mereology are dealing with questions of restricted or unrestricted fusion, supplementation principles or atomicity. The universe of discourse is deciding which axioms can be claimed. A good overview about different mereological systems is given in [Rid 2002] or [Her 2007].

There are two important kinds of application areas of mereology. The first one is an alternative construction of classical set theory and therefore an alternative foundation of mathematics (abstract mereology). The second one aim to an adequate description of the

[1] A translated version is given in [Graes 2003].

[2] He called his theory *mereology*.

[3] The antinomy was discovered by Bertrand Russell in 1901. The consideration of the set of all sets that are not members of themselves leads to a paradox in naive set theory.

structure of the real world, i.e. theories of space, time or material entities (domain-specific mereology).

3.5 Mereotopology

Topology is a mathematical theory build on set theory, which can be understood as an extension of classical geometry. The main task is the investigation of topological spaces and their classification by homeomorphisms. Today it is a sophisticated field with applications to almost every mathematical domain. The proof of the famous *Poincaré conjecture*[4] by Grigori Perelman in 2002/03 represents the latest milestone in this area.

Mereotopology combines mereological theories with topological notions like boundary, closure operator, connection, (external) contact, interior and tangential part and the distinction in open and closed entities. There are several mereotopological theories and they differ in their underlying mereological system and their additional topological primitive predicates. The research work is divided into two fields: 1. Reconstruction of classical geometry based on mereological systems[5] 2. A theory adequate to cognition of the surrounding space and material entities which reflects and explains topological aspects like connection and boundaries according to the realm of experience[6].

Tarskis theory of geometry [Tar 1956] has a model in three-dimensional Euclidean geometry and vice versa. Furthermore he could show that his system is categorical. The second field of investigation is the purpose of this thesis. The research work is at its beginning. The paper of Smith and Varzi [Smi, Var 2000], that we will discuss in chapter 7, is one contribution to this field.

[4]The Poincaré conjecture (every simply connected, boundaryless compact 3-manifold is homeomorphic to the 3-sphere) was one of the seven Prize Problems (1,000,000 dollar) which were advertised due to the Millennium. Grigori Perelman solved this problem but till now he did not publish his proof in a professional journal, what is a condition to get the price money.

[5]That means these theories try to reconstruct the classical geometry not with points as basical entities but rather with extended objects like spheres (or others) and the primitive relation "part of".

[6]Note that the reality is the only model of such a theory. Additional axioms like the 1. Brentanian Thesis (Dependency of zero-, one- and two-dimensional entities) have to be claimed.

Chapter 4

Spatial Entities - The Brentanoraum

4.1 Preliminary

The following theory of the surrounding space is a theory in first-order logic with equality enriched by four primitive relations. Note that we are able to define equality in the theory (see 4.5.1 Identity Principles). The axioms and definitions about space are inspired by the ideas of Brentano [Bren 1976] and Chisholm [Chis 1984]. That is the reason why we will introduce the term "Brentanoraum". The definitions, axioms and propositions are the fundament to describe material entities because they occupy spatial entities.

The universe of discourse of our theory is divided in four classes, namely *space*, *surface*, *line* and *point regions*. Space regions can be understood as compact three-dimensional manifolds which are embeddable into $\mathbf{R}^3$. The most important kind of space regions are connected space regions, so-called *topoids* and we assume that every space region is a finite sum of topoids. Almost all occupied space regions of material objects are topoids, e.g. the occupied space of a car, a chair or a cup. Space region is a basic relation and topoids will be defined.

The other spatial entities are spatial boundaries but note that they have no independent existence (1. Brentanian Thesis)[1]. A spatial boundary of a space region is a two-dimensional boundary (surface region) and we assume that every space region has a maximal boundary. Analogously, surface regions may have one-dimensional boundaries which

[1]Note that the 1. Brentanian Thesis is not true for extraordinary spatial surfaces. We will explain this issue in subsection 4.4.4

are called line regions and line regions themselves may have zero-dimensional boundaries which are called point regions. The two-dimensional and one-dimensional sphere, S^2 and S^1, are examples for spatial entities without boundaries. Surface and line regions correspond to two- or one-dimensional manifolds and if they are connected they are called *surfaces* or *lines*. Note that space, surface and line regions are entities *sui generis*, that means higher-dimensional entities are not a set of lower-dimensional entities.

To describe the structure of space we employ the basic relations *spatial part-of*, *spatial boundary-of* and *coincidence of boundaries*. The parthood relation may be understood as a partial ordering (reflexive, antisymmetric, transitive) on every class of spatial entities. If a spatial entity x is a spatial part of a spatial entity y, then we assume that x has the same dimension as y. This implies that a surface cannot be a spatial part of a space region. For this situation we will define another relation which is called *hyper-part-of*. To capture the situation between a boundary and the spatial entity which is bound by it we use the basic relation spatial boundary-of. The coincidence relation is a binary relation only between same dimensional boundaries. It divides the universe of boundaries in equivalence classes. We use this relation in the sense of two boundaries are located at same place. This relation is important to define contact between spatial entities (2. Brentanian Thesis).

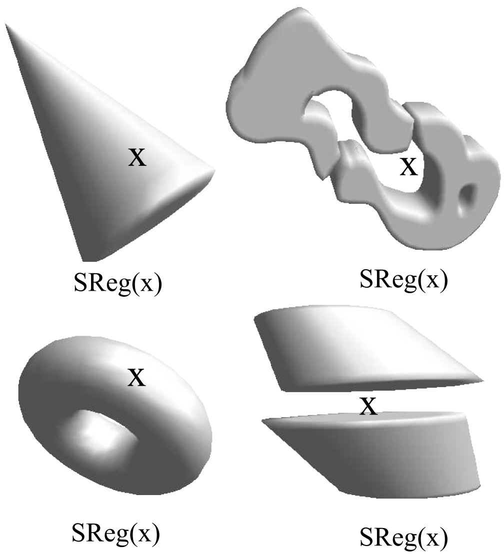


Figure 4.1: Space Regions

4.2 Basic Relations

B1. *SReg(x)* "x is a space region"

B2. *spart(x,y)* "x is a spatial part of y"

B3. *scoinc(x,y)* "x and y are coincident"

B4. *sb(x,y)* "x is a spatial boundary of y"

4.3 Definitions

4.3.1 Standard Definitions

D1. $sppart(x,y) \Leftrightarrow spart(x,y) \wedge x \neq y$ "x is a spatial proper part of y"

D2. $sov(x,y) \Leftrightarrow \exists z\ (spart(z,x) \wedge spart(z,y))$ "spatial overlap of space regions"

D3. $sum_n(x_1, ..., x_n) = x \Leftrightarrow \forall x`(sov(x`, x) \leftrightarrow \bigvee_{i=1}^{n} sov(x`, x_i))$

"mereological sum of x_1,...,x_n"

D3‘. $sum(x,y)=z \Leftrightarrow \forall w\ (sov(w,z) \leftrightarrow sov(w,x) \vee sov(w,y))$

"mereological sum of x and y "

D4. $intersect_n(x_1, ..., x_n) = x \Leftrightarrow \forall x`(spart(x`, x) \leftrightarrow \bigwedge_{i=1}^{n} spart(x`, x_i))$

"mereological intersection of x_1,...,x_n"

D4‘. $intersect(x,y)=z \Leftrightarrow \forall w\ (spart(w,z) \leftrightarrow spart(w,x) \wedge spart(w,y))$

"mereological intersection of x and y "

D5. $relcompl_n(x_1, ..., x_n) = x \Leftrightarrow \forall x`(spart(x`, x) \leftrightarrow \bigwedge_{i=1}^{n-1} \neg sov(x`, x_i) \wedge spart(x`, x_n))$

"relative complement of x_n and x_1,...,x_{n-1}"

D5‘. $relcompl(x,y)=z \Leftrightarrow \forall w\ (spart(w,z) \leftrightarrow \neg sov(w,x) \wedge spart(w,y))$

"relative complement of x and y (z=y-x)"

Note that the definitions of the mereological functions are schemata of definitions. The most important cases are the definitions D3‘, D4‘ and D5‘ with two arguments. In T5, T6 and T7 (see subsection 4.5.2) we will show the uniqueness of these relations, that means the mereological sum, intersection and complement are in fact functions.

Figures

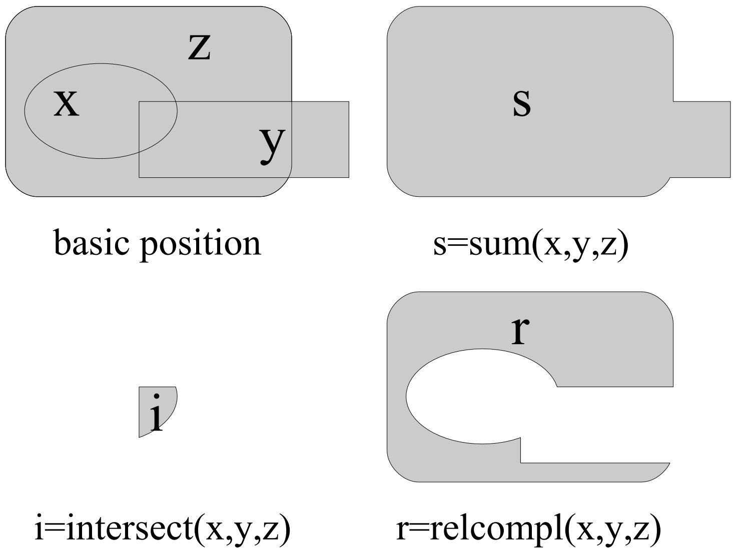


Figure 4.2: Mereological Functions

4.3.2 Spatial Boundaries

D6. *2DB(x)* $\Leftrightarrow \exists y$ *(SReg(y)* $\wedge$ *sb(x,y))*

”x is a 2-dimensional boundary (surface region)”

D7. *1DB(x)* $\Leftrightarrow \exists y$ *(2DB(y)* $\wedge$ *sb(x,y))*

”x is a 1-dimensional boundary (line region)”

D8. *0DB(x)* $\Leftrightarrow \exists y$ *(1DB(y)* $\wedge$ *sb(x,y))*

”x is a 0-dimensional boundary (point region)”

D9. $SB(x) \Leftrightarrow \exists y\ sb(x,y)$ "x is a spatial boundary"

D10. $maxb(x,y) \Leftrightarrow sb(x,y) \wedge \forall z\ (sb(z,y) \rightarrow spart(z,x))$ "x is maximal boundary of y"

D11. $MaxB(x)=y \Leftrightarrow maxb(y,x)$ "maximal boundary function"

In axiom A20 we will claim the conditional existence for spatial entitities and in theorem T8 we will prove the uniqueness. That means "MaxB(x)" is in fact a function.

D12. $2db(x,y) \Leftrightarrow (SReg(y) \wedge sb(x,y))$

"x is a 2-dimensional boundary (surface region) of y"

D13. $1db(x,y) \Leftrightarrow (2DB(y) \wedge sb(x,y))$

"x is a 1-dimensional boundary (line region) of y"

D14. $0db(x,y) \Leftrightarrow (1DB(y) \wedge sb(x,y))$

"x is a 0-dimensional boundary (point region) of y"

Figures

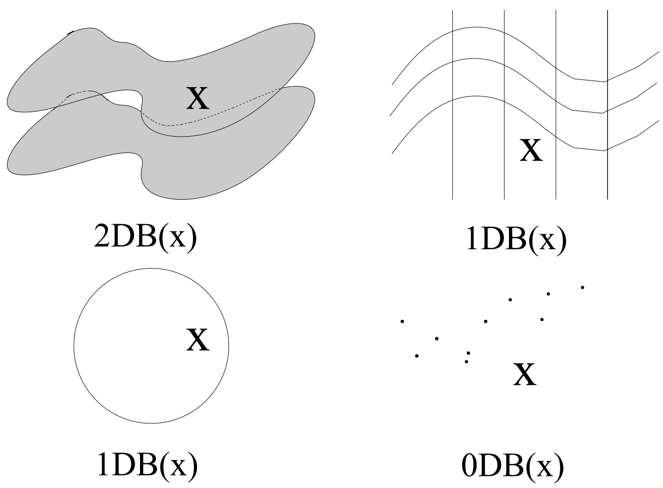


Figure 4.3: Spatial Boundaries

4.3.3 Ordinary and Extraordinary Spatial Entities

Spatial Entities can be divided in four classes: space regions, surface regions, line regions and point regions. Furthermore we want to distinguish between ordinary and extraordinary entities. Consider two different cubes x and y which lie upon each other. The upper side of cube x and the lower side of cube y occupy two different spatial surfaces which are coincident. Consider now the mereological sum of both surfaces ("double surface"). This sum is an example for an extraordinary spatial entity.

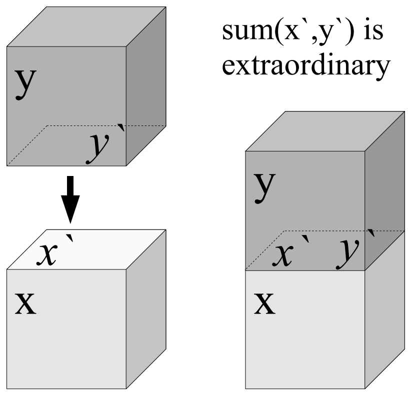


Figure 4.4: Extraordinariness (Cubes)

Ordinary entities are not only theoretical objects which result from mereological functions. Imagine therefor a lying solid rubber sleeve which is cut through vertical at one position (both "ends" are in contact). The maximal material boundary of this object occupy an extraordinary spatial boundary[2] because there is a "double-surface" at the cutting site.

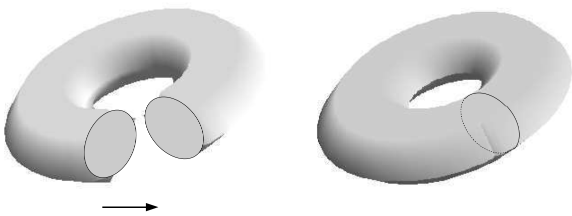


Figure 4.5: Extraordinariness (Torus)

[2]A detailed consideration of the interrelations between spatial and material entities is presented in subsection 5.3.4.

Note that three-dimensional spatial entities are ordinary per definition.

D15. $Ord(x) \Leftrightarrow \neg\exists x`x``(spart(x`,x) \wedge spart(x``,x) \wedge \neg sov(x`,x``) \wedge scoinc(x`,x``))$

"x is a ordinary spatial entity"

D16. $ExOrd(x) \Leftrightarrow \neg Ord(x)$

"x is a extraordinary spatial entity"

4.3.4 Hyper Parts

The term "hyper part" is according to the mathematical concept "hyperplane", which circumstantiate a subspace with co-dimension 1. We will use it for parts with co-dimension greater than or equal to 1.

D17. $2dhypp(x,y) \Leftrightarrow \exists z\ (spart(z,y) \wedge 2db(x,z))$

"x is a 2-dimensional hyper part of y"

D18. $1dhypp(x,y) \Leftrightarrow \exists z\ (spart(z,y) \wedge 1db(x,z)) \vee \exists z\ (2dhypp(z,y) \wedge 1db(x,z))$

"x is a 1-dimensional hyper part of y"

D19. $0dhypp(x,y) \Leftrightarrow \exists z\ (spart(z,y) \wedge 0db(x,z)) \vee \exists z\ (1dhypp(z,y) \wedge 0db(x,z))$

"x is a 0-dimensional hyper part of y"

D20. $hypp(x,y) \Leftrightarrow 2dhypp(x,y) \vee 1dhypp(x,y) \vee 0dhypp(x,y)$

"x is a hyper part of y"

Figures

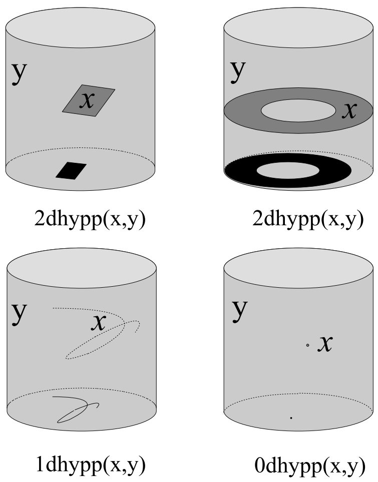


Figure 4.6: Hyper Parts (Cylinder)

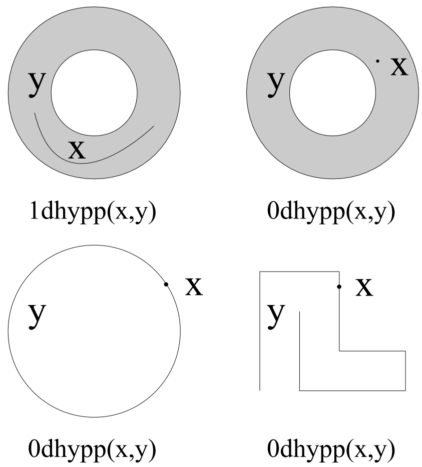


Figure 4.7: Hyper Parts (Annulus, Lines)

4.3.5 Inner and Tangential Parts

Inner parts are spatial parts of an entity x which are not connected with the maximal boundary of x. A tangential part is a spatial part which is not an inner part. The following question arises: What are spatial parts of an boundaryless entity? That is a question of belief and we decided to call these parts inner parts.

Inner Parts

D21. $inpart(x,y) \Leftrightarrow spart(x,y) \wedge (\neg\exists MaxB(y) \vee (\exists z\ (maxb(z,y) \wedge \forall uv\ (hypp(u,x) \wedge (hypp(v,z) \vee spart(v,z)) \rightarrow \neg scoinc(u,v))))$

"x is a (equal dimensional) inner part of y"

D22. $2dhypinpart(x,y) \Leftrightarrow \exists z\ (inpart(z,y) \wedge 2dhypp(x,z))$

"x is a two-dimensional hyper inpart of y"

D23. $1dhypinpart(x,y) \Leftrightarrow \exists z\ (inpart(z,y) \wedge 1dhypp(x,z))$

"x is a one-dimensional hyper inpart of y"

D24. *0dhypinpart(x,y)* $\Leftrightarrow$ $\exists z$ *(inpart(z,y)* $\wedge$ *0dhypp(x,z))*

"x is a zero-dimensional hyper inpart of y"

D25. *hypinpart(x,y)* $\Leftrightarrow$ *2dhypinpart(x,y)* $\vee$ *1dhypinpart(x,y)* $\vee$ *0Dhypinpart(x,y)*

"x is a hyper inpart of y"

Figures

Note that all figures except the annulus inside the cylinder (top right figure 4.6) in subsection 4.3.4 Hyper Parts are examples of hyper inner parts too. That is why we want to give only a few more interesting examples.

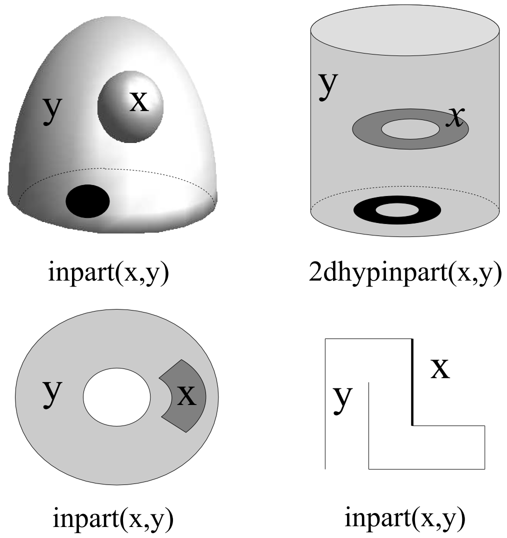


Figure 4.8: Inner Parts Examples

Tangential Parts

D26. *tangpart(x,y)* $\Leftrightarrow$ *spart(x,y)* $\wedge$ $\neg$*inpart(x,y)*

"x is a tangential part of y"

D27. *2dhyptangpart(x,y)* $\Leftrightarrow$ $\neg$*2dhypinpart(x,y)* $\wedge$ *2dhypp(x,y)*

"x is a two-dimensional hyper tangential part of y"

D28. $\mathit{1dhyptangpart}(x,y) \Leftrightarrow \neg \mathit{1dhypinpart}(x,y) \wedge \mathit{1dhypp}(x,y)$

"x is a one-dimensional hyper tangential part of y"

D29. $\mathit{0dhyptangpart}(x,y) \Leftrightarrow \neg \mathit{0dhypinpart}(x,y) \wedge \mathit{0dhypp}(x,y)$

"x is a zero-dimensional hyper tangential part of y"

D30. $\mathit{hyptangpart}(x,y) \Leftrightarrow \mathit{2dhyptangpart}(x,y) \vee \mathit{1dhyptangpart}(x,y) \vee \mathit{0Dhyptangpart}(x,y)$

"x is a hyper tangential part of y"

Figures

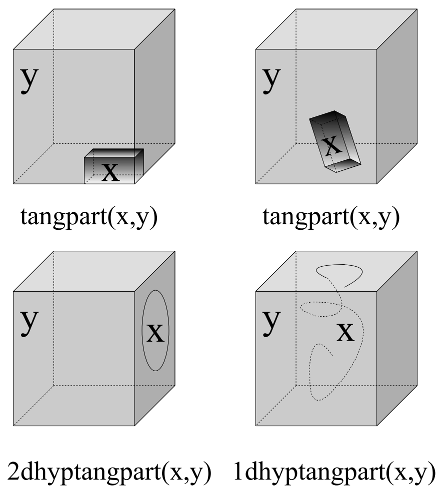


Figure 4.9: Tangential Parts (Cube)

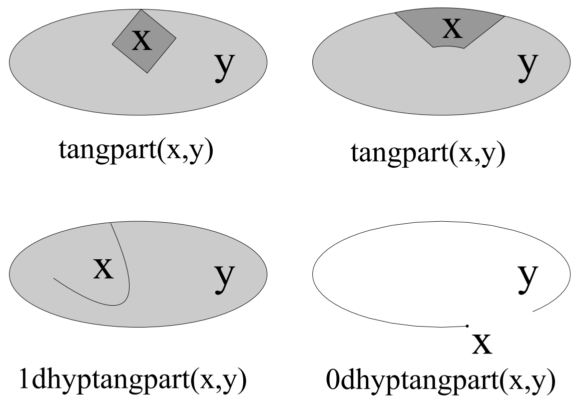


Figure 4.10: Tangential Parts (Ellipse, Line)

4.3.6 Connected Entities

The mathematical concept of connection requires a definition of open respective closed spaces. Our definitions are aimed at a description of material entities (adequate to cognition) and we assume that there are just no "open" material entities in reality (compare subsection 3.3.1). Keep in mind that it is possible to define open and closed entities.

D31. $\mathit{eqdim}(x,y) \Leftrightarrow (\mathit{SReg}(x) \wedge \mathit{SReg}(y)) \vee (\mathit{2DB}(x) \wedge \mathit{2DB}(y)) \vee (\mathit{1DB}(x) \wedge \mathit{1DB}(y)) \vee (\mathit{0DB}(x) \wedge \mathit{0DB}(y))$

"equal dimension"

Spatial Connectedness

D32. $2DC(x) \Leftrightarrow SReg(x) \wedge \neg\exists yz\ (eqdim(y,z) \wedge sum(y,z)=x \wedge \neg sov(y,z) \wedge \forall y\acute{}z\acute{}\ (2db(y\acute{},y) \wedge 2db(z\acute{},z) \rightarrow \neg\ scoinc(y\acute{},z\acute{})))$

"x is 2-dimensional connected"

The following definition is semantically equivalent to the definition above. This alternative definition is useful to prove that two-dimensional connectedness implies one-dimensional connectedness (see subsection 4.5.4).

D32´. $2DC(x) \Leftrightarrow SReg(x) \wedge \forall yz\ (\neg eqdim(y,z) \vee \neg sum(y,z)=x \vee sov(y,z) \vee \exists y\acute{}z\acute{}\ (2db(y\acute{},y) \wedge 2db(z\acute{},z) \wedge scoinc(y\acute{},z\acute{})))$

"x is 2-dimensional connected"

D33. $1DC(x) \Leftrightarrow (SReg(x) \vee 2DB(x)) \wedge \neg\exists yz\ (eqdim(y,z) \wedge sum(y,z)=x \wedge \neg sov(y,z) \wedge \forall y\acute{}z\acute{}\ (1dhypp(y\acute{},y) \wedge 1dhypp(z\acute{},z) \rightarrow \neg\ scoinc(y\acute{},z\acute{})))$

"x is 1-dimensional connected"

The following semantical equivalent definition is useful to prove that one-dimensional connectedness implies zero-dimensional connectedness (see subsection 4.5.4).

D33´. $1DC(x) \Leftrightarrow (SReg(x) \vee 2DB(x)) \wedge \forall yz\ (\neg eqdim(y,z) \vee \neg sum(y,z)=x \vee sov(y,z) \vee \exists y\acute{}z\acute{}\ (1dhypp(y\acute{},y) \wedge 1dhypp(z\acute{},z) \wedge scoinc(y\acute{},z\acute{})))$

"x is 1-dimensional connected"

D34. $0DC(x) \Leftrightarrow (SReg(x) \vee 2DB(x) \vee 1DB(x))) \wedge \neg\exists yz\ (eqdim(y,z) \wedge sum(y,z)=x \wedge \neg sov(y,z) \wedge \forall y\acute{}z\acute{}\ (0dhypp(y\acute{},y) \wedge 0dhypp(z\acute{},z) \rightarrow \neg\ scoinc(y\acute{},z\acute{})))$

"x is 0-dimensional connected"

The following definition will be used in subsection 4.5.7 for the theorem T44 (an ordinary line only may coincident with ordinary lines).

D34´. $0DC(x) \Leftrightarrow (SReg(x) \vee 2DB(x) \vee 1DB(x)) \wedge \forall yz\ (\neg eqdim(y,z) \vee \neg sum(y,z)=x \vee sov(y,z) \vee \exists y\acute{}z\acute{}\ (0dhypp(y\acute{},y) \wedge 0dhypp(z\acute{},z) \wedge scoinc(y\acute{},z\acute{})))$

"x is 0-dimensional connected"

D35. *C(x)* $\Leftrightarrow$ *2DC(x)* $\vee$ *1DC(x)* $\vee$ *0DC(x)* $\vee$ *0D(x)* "x is connected"

D36. *c(x,y)* $\Leftrightarrow$ *C(sum(x,y))* "x and y are connected"

D37. *exc(x,y)* $\Leftrightarrow$ *c(x,y)* $\wedge$ $\neg$*sov(x,y)* "x and y are external connected"

The definition of external contact applies to all spatial entities. The following question arises: What is meant by external contact for two- or one-dimensional entities? Is it an external contact if two non-overlapping lines or surfaces interpenetrate, that means is there a n-crosspoint or a n-crossline (for definition see subsection 4.3.8) at the interpenetration area? Our definition says yes. If necessary, one can give these situations another name, e.g. "interpenetration(x,y)". This phenomenon arise, because of the embedding into the three-dimensional space. Interpenetration and spatial overlap is the same for three-dimensional entities but not for surfaces or lines. The notion of external contact for three-, two- or one-dimensional entities is the same if we restrict the dimension of the embedding space to three, two or one.

Classification by Connectedness

With the help of the definitions above we can distinguish several cases of connectedness. There are three kinds of three-dimensional connected entities. The most important one is the so-called *topoid*.

D38. *Top(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ *2DC(x)* "x is a topoid (2d-connected space region)"

Note that we excluded three-dimensional entities which are connected by a line or a point, although they are connected spatial entities. These situations are captured in the following two definitions.

D39. *Top1DC(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ *1DC(x)* $\wedge$ $\neg$*2DC(x)*

"x is a quasi topoid (1d-connected space region)"

D40. *Top0DC(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ *0DC(x)* $\wedge$ $\neg$*1DC(x)*

"x is a quasi topoid (0d-connected space region)"

For two-dimensional connected entities we have to distinguish between two cases.

D41. *2D(x)* $\Leftrightarrow$ *2DB(x)* $\wedge$ *1DC(x)* ”x is a 1d-connected surface”

D42. *2D0DC(x)* $\Leftrightarrow$ *2DB(x)* $\wedge$ *0DC(x)* $\wedge$ $\neg$*1DC(x)* ”x is a 0d-connected surface”

There is only one type of connectedness for one-dimensional entities.

D43. *1D(x)* $\Leftrightarrow$ *1DB(x)* $\wedge$ *0DC(x)* ”x is a 0d-connected line”

D44. *0D(x)* $\Leftrightarrow$ *0DB(x)* $\wedge$ $\neg\exists$*y sppart(y,x)* ”x is a point”

Figures

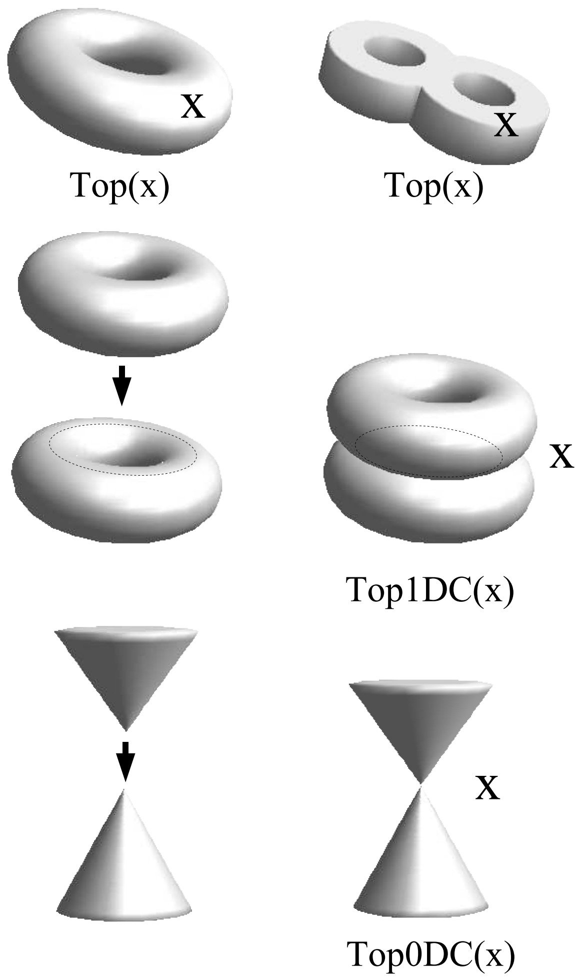


Figure 4.11: Three-Dimensional Connected Entities

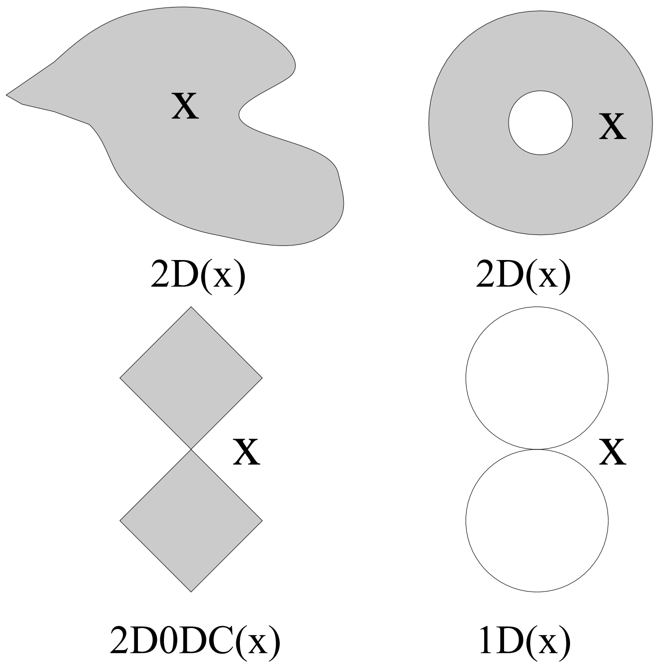


Figure 4.12: Two- and One-Dimensional Connected Entities

4.3.7 Touching Areas

The occupied topoids of a palm and a table are extern connected if you put your palm on a table. We want to distinguish between the touching area of the topoid$_{palm}$ and the topoid$_{table}$ and the touching area of the topoid$_{table}$ and the topoid$_{palm}$. That means our definition of a touching area implies a notion of belonging. The two-dimensional touching area of the topoid$_{palm}$ and the topoid$_{table}$ "belongs" (two-dimensional boundary) to the topoid$_{palm}$ and vice versa. By choosing this definition the touching area relation is not symmetric (toucharea(x,y)$\neq$toucharea(y,x)) but we can show that for every touching area of x and y exists a coincident touching area of y and x (see section 4.5.6).

Note that it is possible to define a symmetric touching area relation, e.g. mereological sum of their coincident boundaries or hyper parts. The price of symmetry is the loss of the notion of belonging. Furthermore in case of two-dimensional touching areas we lost the ordinariness of these entities which we will show in subsection 4.5.6.

There are several problems when dealing with maximal touching areas. In case of maximal two-dimensional touching areas we will claim (see axiom A24) the existence if there are a two-dimensional touching area. Note that this axiom does not hold for one- or zero-dimensional touching areas because they need not to be ordinary (see figure 4.13). In case of the existence of a certain maximal touching area we will show the uniqueness of it (compare theorem T9).

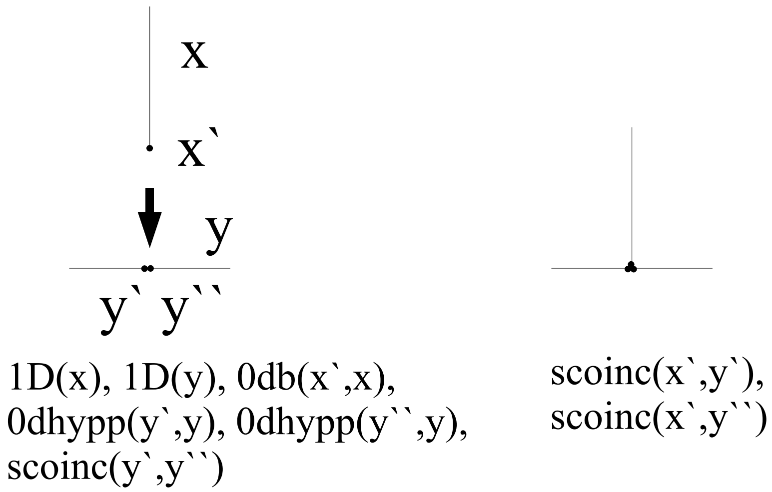


Figure 4.13: No Maximal Touching Area (Lines)

The figure above shows that y has two zero-dimensional touching areas, namely y‘ and y“ which coincide with x‘, but there is no maximal touching area of y. Assuming that the mereological sum of y‘ and y“ is the maximal touching area of y leads to a contradiction, if we claim that coincidence is only possible between two ordinary or two extraordinary entities (compare definitions D15/16). We will assume this because we want to understand coincidence as a relation between similar entities.

Another problem is the possibility of touching areas which consist of different dimensional entities, e.g. a touching area as mereological sum of a line and a point (see figure 4.14). We want to exclude such kinds of entities and only distinguish between two-, one-, zero-dimensional touching areas (future work).

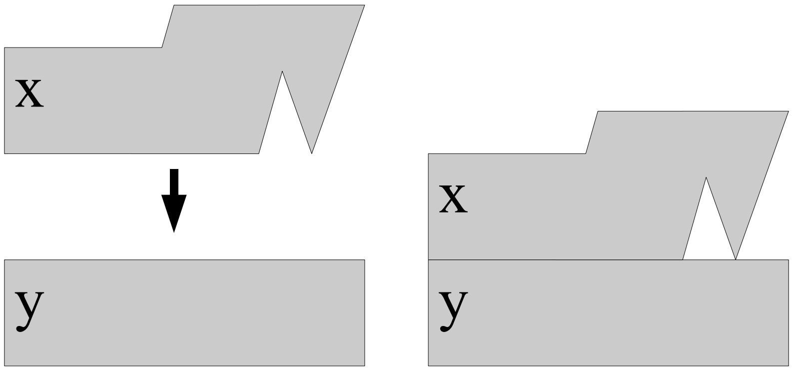


Figure 4.14: Touching Area with Mixed Dimension (Line and Point)

D45. *2dtoucharea(x,y,z)* $\Leftrightarrow$ *exc(y,z)* $\wedge$ *2dhypp(x,y)* $\wedge$ $\exists u$ *(u$\neq$x* $\wedge$ *2dhypp(u,z)* $\wedge$ *sco-inc(x,u))*

"two-dimensional touching area relation"

D46. *1dtoucharea(x,y,z)* $\Leftrightarrow$ *exc(y,z)* $\wedge$ *1dhypp(x,y)* $\wedge$ $\exists u$ *(u$\neq$x* $\wedge$ *1dhypp(u,z)* $\wedge$ *sco-inc(x,u))*

"one-dimensional touching area relation"

D47. *0dtoucharea(x,y,z)* $\Leftrightarrow$ *exc(y,z)* $\wedge$ *0dhypp(x,y)* $\wedge$ $\exists u$ *(u$\neq$x* $\wedge$ *0dhypp(u,z)* $\wedge$ *sco-inc(x,u))*

"zero-dimensional touching area relation"

D48. *toucharea(x,y,z)* $\Leftrightarrow$ *2dtoucharea(x,y,z)* $\vee$ *1dtoucharea(x,y,z)* $\vee$ *0dtoucharea(x,y,z))*

"touching area relation"

D49. *2DTouchArea(y,z)=x* $\Leftrightarrow$ *2dtoucharea(x,y,z)*

"x is a two-dimensional touching area of y and z"

D50. *1DTouchArea(y,z)=x* $\Leftrightarrow$ *1dtoucharea(x,y,z)*

"x is a one-dimensional touching area of y and z"

D51. *0DTouchArea(y,z)=x* $\Leftrightarrow$ *0dtoucharea(x,y,z)*

"x is a zero- dimensional touching area of y and z"

D52. $\mathit{Max2DTouchArea}(y,z)=x \Leftrightarrow \mathit{2dtoucharea}(x,y,z) \land \forall x' \; (\mathit{2dtoucharea}(x',y,z) \rightarrow \mathit{spart}(x',x))$

"maximal two-dimensional touching area function"

D53. $\mathit{Max1DTouchArea}(y,z)=x \Leftrightarrow \mathit{1dtoucharea}(x,y,z) \land \forall x' \; (\mathit{1dtoucharea}(x',y,z) \rightarrow \mathit{spart}(x',x))$

"maximal one-dimensional touching area function"

D54. $\mathit{Max0DTouchArea}(y,z)=x \Leftrightarrow \mathit{0dtoucharea}(x,y,z) \land \forall x' \; (\mathit{0dtoucharea}(x',y,z) \rightarrow \mathit{spart}(x',x))$

"maximal zero-dimensional touching area function"

D55. $\mathit{MaxTouchArea}(y,z)=x \Leftrightarrow \mathit{Max2DTouchArea}(y,z)=x \lor \mathit{Max1DTouchArea}(y,z)=x \lor \mathit{Max0DTouchArea}(y,z)=x$

"maximal touching area function"

In subsection 4.5.2 we will show that the maximal touching area is in fact a function. Note that the existence of a one- or zero-dimensional touching area does not imply the existence of a maximal touching area but in case of yes, the existence is unique.

Figures

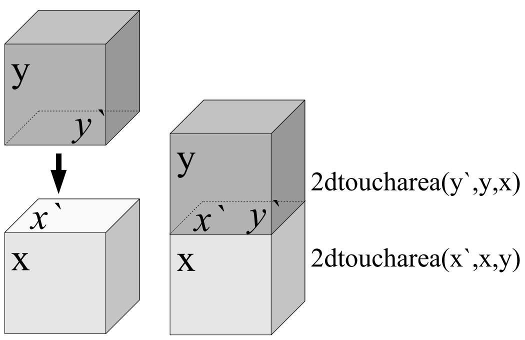


Figure 4.15: Touching Area (Cubes)

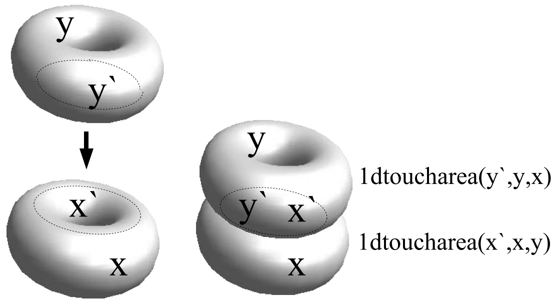


Figure 4.16: Touching Area (Tori)

4.3.8 Cross-Entities

The following spatial entities are special kinds of extraordinary entities (in case of n≥2). We will call them n-crosspoints, n-crosslines or n-crosssurfaces because they usually appear if two spatial entities interpenetrate or cross each other. The "n" stands for a n-fold non-overlapping division with certain properties of the cross-entity and we will call the "n" of a n-cross-entity x the cardinality of x.

Imagine a five-way crossing and further that the streets are lines (see figure 4.17). Every line x_i has an ending point x_i‘ (at the crossroad). All ending points are pairwise distinct and coincide with each other. The mereological sum of all ending points (x= sum(x_1‘,x_1‘,x_1‘,x_1‘,x_5‘)) is an example of a 5-crosspoint. In subsection 4.5.7 we will show that a n-crosspoint is no m-crosspoint for n≠m and furthermore if a n-crosspoint x is coincident with a m-crosspoint y, then it must be n=m.

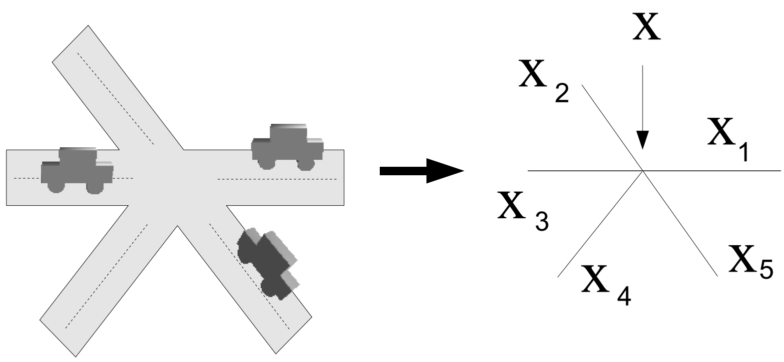


Figure 4.17: Crosspoint Example

Crosslines and crosssurfaces are the higher-dimensional analog to crosspoints. In theorem T30 we will show that there are no three surfaces which coincide with each other. That is why there are only two kinds of crosssurfaces, namely the ordinary 1-crosssurface and the extraordinary 2-crosssurface (see figure 4.4 and 4.5).

Note that the following definitions are schemata of definitions.

D56. $equ(x_1,...,x_n,x_1`,...x_n`) \Leftrightarrow (\bigwedge_{1\le i<j\le n} x_i \neq x_j \wedge\ x_i` \neq x_j`) \wedge ((x_1 = x_1` \wedge ... \wedge x_n = x_n`) \vee (x_1 = x_1` \wedge ... \wedge x_{n-2} = x_{n-2}` \wedge x_{n-1} = x_n` \wedge x_n = x_{n-1}`) \vee ... \vee (x_1 = x_n` \wedge x_2 = x_{n-1}` \wedge ... \wedge x_n = x_1`))$

"pairwise equality"

The second conjunction describes the n! (n-factorial) possibilities of pairwise equality. The set theoretical notation of definition D56 is $\{x_1,...,x_n\}$=$\{x_1`,...,x_n`\}$ and $|\{x_1,...,x_n\}|$=n.

D57. $Cross0DB_n(x) \Leftrightarrow \exists x_1...x_n(x = sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^n 0D(x_i)) \wedge (\bigwedge_{1\le i<j\le n} x_i \neq x_j \wedge scoinc(x_i,x_j)))$

"x is a n-crosspoint"

The definitions of a n-crossline and a n-crosssurface are longer than the definition of a n-crosspoint. We have to guarantee that a n-crossline is no m-crossline for n≠m. In case of n-crosspoints we will show this property[3] (see theorem T39).

D58. $Cross1DB_n(x) \Leftrightarrow \exists x_1...x_n(x = sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^n 1D(x_i) \wedge Ord(x_i)) \wedge (\bigwedge_{1\le i<j\le n} \neg sov(x_i,x_j) \wedge\ scoinc(x_i,x_j))) \wedge (\bigwedge_{i=1}^{n-1}(\neg\exists x_1`...x_i`(x = sum(x_1`,...,x_i`) \wedge (\bigwedge_{k=1}^i 1D(x_k`) \wedge Ord(x_k`)) \wedge\ (\bigwedge_{1\le k<l\le i} \neg sov(x_k`,x_l`) \wedge\ scoinc(x_k`,x_l`)))))$

"x is a n-crossline"

D59. $Cross2DB_n(x) \Leftrightarrow \exists x_1...x_n(x = sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^n 2D(x_i) \wedge Ord(x_i)) \wedge (\bigwedge_{1\le i<j\le n} \neg sov(x_i,x_j) \wedge\ scoinc(x_i,x_j))) \wedge (\bigwedge_{i=1}^{n-1}(\neg\exists x_1`...x_i`(x = sum(x_1`,...,x_i`) \wedge (\bigwedge_{k=1}^i 2D(x_k`) \wedge Ord(x_k`)) \wedge\ (\bigwedge_{1\le k<l\le i} \neg sov(x_k`,x_l`) \wedge\ scoinc(x_k`,x_l`)))))$

"x is a n-crosssurface"

D60. $cross0db_n(x,y) \Leftrightarrow \exists x_1...x_n y_1...y_n(x = sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^n 0D(x_i) \wedge 1D(y_i) \wedge spart(y_i,y) \wedge sb(x_i,y_i)) \wedge\ (\bigwedge_{1\le i<j\le n} \neg sov(y_i,y_j) \wedge scoinc(x_i,x_j) \wedge x_i \neq x_j))$

"x is a n-crosspoint of y"

D61. $cross1db_n(x,y) \Leftrightarrow \exists x_1...x_n y_1...y_n(x = sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^n 1D(x_i) \wedge 2D(y_i) \wedge Ord(x_i) \wedge spart(y_i,y) \wedge sb(x_i,y_i)) \wedge\ (\bigwedge_{1\le i<j\le n} \neg sov(y_i,y_j) \wedge scoinc(x_i,x_j) \wedge \neg sov(x_i,x_j))) \wedge (\bigwedge_{i=1}^{n-1}(\neg\exists x_1`...x_i`y_1`...y_i`(x = sum(x_1`,...,x_i`) \wedge (\bigwedge_{k=1}^i 1D(x_k`) \wedge 2D(y_k`) \wedge Ord(x_k`) \wedge spart(y_k`,y) \wedge sb(x_k`,y_k`)) \wedge\ (\bigwedge_{1\le k<l\le i} \neg sov(y_k`,y_l`) \wedge scoinc(x_k`,x_l`) \wedge \neg sov(x_k`,x_l`)))))$

[3]Crosspoints are the only cross-entities that are "built" of atoms (points). That means the "building blocks" of a n-crosspoint are unique in contrast to n-crosslines or -surfaces. That is why we can show this property only for crosspoints.

"x is a n-crossline of y"

D62. $cross2db_n(x,y) \Leftrightarrow \exists x_1...x_n y_1...y_n(x = sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^{n} 2D(x_i) \wedge Top(y_i) \wedge Ord(x_i) \wedge spart(y_i,y) \wedge sb(x_i,y_i)) \wedge (\bigwedge_{1\leq i<j\leq n} \neg sov(y_i,y_j) \wedge scoinc(x_i,x_j) \wedge \neg sov(x_i,x_j))) \wedge (\bigwedge_{i=1}^{n-1}(\neg\exists x_1`...x_i`y_1`...y_i`(x = sum(x_1`,...,x_i`) \wedge (\bigwedge_{k=1}^{i} 2D(x_k`) \wedge Top(y_k) \wedge Ord(x_k`) \wedge spart(y_k`,y) \wedge sb(x_k`,y_k`)) \wedge (\bigwedge_{1\leq k<l\leq i} \neg sov(y_k`,y_l`) \wedge scoinc(x_k`,x_l`) \wedge \neg sov(x_k`,x_l`))))))$

"x is a n-crosssurface of y"

4.3.9 Some Remarks about Generalization

Mixed Dimension

The universe of discourse of the Brentanoraum is divided into four classes, namely space, surface, line and point regions. Space entities with a mixed dimension are excluded, e.g. the mereological sum of a line and a point (see figure 4.14). Note that space entities which consist of different dimensional entities are not only an abstract construction, what we have seen in subsection 4.3.7 (touching areas).

Here we want to give some proposals to generalize the standard mereological functions but note that these functions or relations are not included in our axiomatization.

D63. $hypsov(x,y) \Leftrightarrow \exists x`y`((hypp(x`,x) \vee spart(x`,x)) \wedge (hypp(y`,y) \vee spart(y`,y)) \wedge scoinc(x`,y`)$

"hyper spatial overlap"

D64. $hypsum_n(x_1,...,x_n) = x \Leftrightarrow \forall x`(hypsov(x`,x) \leftrightarrow \bigvee_{i=1}^{n} hypsov(x`,x_i))$

"hyper mereological sum of $x_1,...,x_n$"

D65. $hypintersect_n(x_1,...,x_n) = x \Leftrightarrow \forall x`(spart(x`,x) \vee hypp(x`,x) \leftrightarrow \bigwedge_{i=1}^{n}(spart(x`,x_i) \vee hypp(x`,x_i)))$

"hyper mereological intersection of $x_1,...,x_n$"

D66. $hyprelcompl_n(x_1,...,x_n) = x \Leftrightarrow \forall x`(spart(x`,x) \vee hypp(x`,x) \leftrightarrow \bigwedge_{i=1}^{n-1}(\neg spart(x`,x_i) \vee \neg hypp(x`,x_i)) \wedge (spart(x`,x_n) \vee hypp(x`,x_n)))$

"hyper mereological intersection of x_n and $x_1,..,x_{n-1}$"

Figures

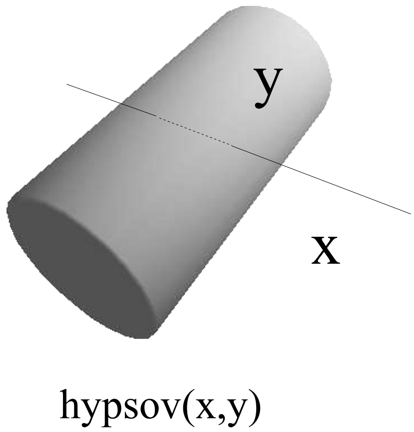


Figure 4.18: Hyper Spatial Overlap (Cylinder and Line)

Connected Components

In subsection 4.3.6 we talked about connected entities and gave a classification of them. What about non-connected entities? We will define three different versions of the term "Connected Components". They differ in their underlying spatial entities ("building blocks") which are counted. We will give these definitions for space regions but note that it is possible to generalize them for lower-dimensional entities[4].

In D32, D33 and D34 we defined three different kinds of connectedness, namely two-, one- and zero-dimensional connected. With the help of them we give the following different definitions of connected components.

D67. *1CC(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ *2DC(x)*

"the cardinality of connected components of x is one"

D67'. *1CC'(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ *1DC(x)*

"the cardinality of connected components of x is one"

[4]One has to take into consideration that lower-dimensional entities may be extraordinary and the question arises how to count these entities. That means there are more possibilities to define connected components for lower-dimensional entities.

D67“. *1CC“(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ *0DC(x)*

”the cardinality of connected components of x is one”

With the help of the definitions above we can define inductively the notion of ”x consists of 2,3,...,n connected components”.

D68. *nCC(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ $(\bigwedge_{i=1}^{n-1} \neg iCC(x)) \wedge \exists x_1 ... x_n (x = sum(x_1, ..., x_n) \wedge$ $(\bigwedge_{i=1}^{n} 1CC(x_i)) \wedge$ $(\bigwedge_{1 \leq i < j \leq n} \neg sov(x_i, x_j)))$

”the cardinality of connected components of x is n”

D68‘. *nCC‘(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ $(\bigwedge_{i=1}^{n-1} \neg iCC‘(x)) \wedge \exists x_1 ... x_n (x = sum(x_1, ..., x_n) \wedge$ $(\bigwedge_{i=1}^{n} 1CC‘(x_i)) \wedge$ $(\bigwedge_{1 \leq i < j \leq n} \neg sov(x_i, x_j)))$

”the cardinality of connected components of x is n”

D68“. *nCC“(x)* $\Leftrightarrow$ *SReg(x)* $\wedge$ $(\bigwedge_{i=1}^{n-1} \neg iCC“(x)) \wedge \exists x_1 ... x_n (x = sum(x_1, ..., x_n) \wedge$ $(\bigwedge_{i=1}^{n} 1CC“(x_i)) \wedge$ $(\bigwedge_{1 \leq i < j \leq n} \neg sov(x_i, x_j)))$

”the cardinality of connected components of x is n”

Figures

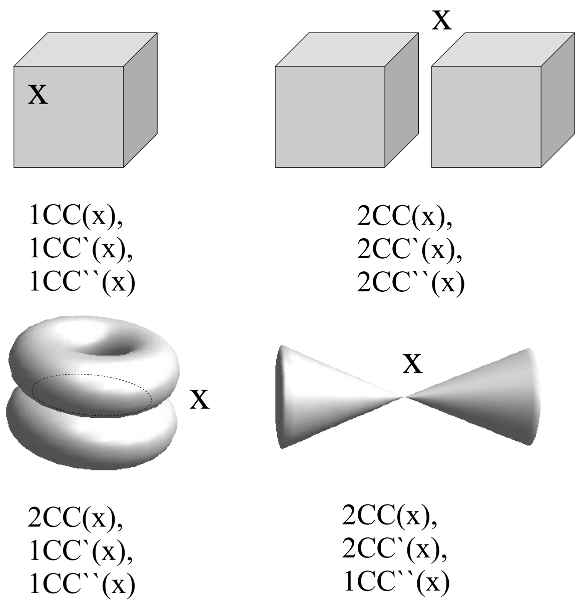


Figure 4.19: Connected Components 1

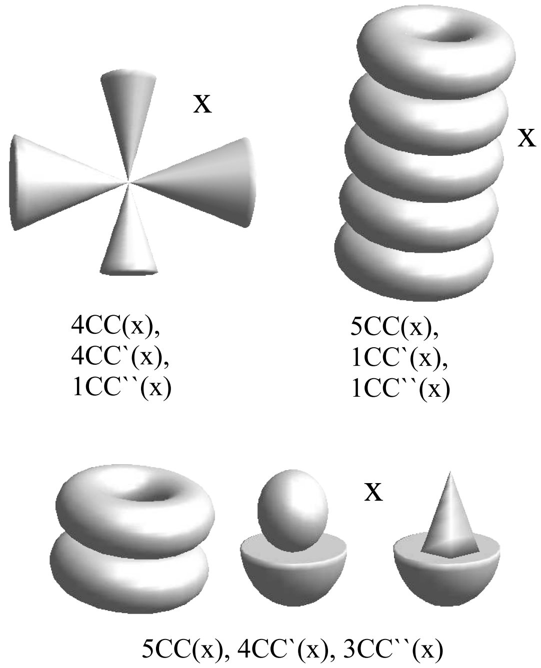


Figure 4.20: Connected Components 2

Because of theorems T24 and T25 (higher-dimensional connectedness implies lower-dimensional connectedness) one may agree that for all space regions x the following inequality is true.

$$\forall x, n, k, l(nCC(x) \wedge kCC'(x) \wedge lCC''(x) \rightarrow n \geq k \geq l) \qquad \text{(CC-inequality)}$$

4.4 Axioms

4.4.1 Mereology of Space

There are a number of different mereological systems which are used to describe the structure of space. We want to introduce the axioms of a so-called *classical extensional mereology (CEM)*.

A1. $\forall x$ *(spart(x,x))* "reflexivity of spatial part"

A2. $\forall xy$ *(spart(x,y)* $\wedge$ *spart(y,x)* $\rightarrow$ *x=y)* "antisymmetry of spatial part"

A3. $\forall xyz$ *(spart(x,y)* $\wedge$ *spart(y,z)* $\rightarrow$ *spart(x,z))* "transitivity of spatial part"

These three axioms describe a minimal system which is called *ground mereology (M)*. The relation *spatial part* satisfies the condition of a partial ordering.

A4. $\forall xy$ *(spart(x,y)* $\rightarrow$ *eqdim(x,y))* "range restriction"

A5. $\forall xy$ *(*$\neg$*spart(y,x)* $\rightarrow$ $\exists z$ *(spart(z,y)* $\wedge$ $\neg$*sov(z,x)))*

"strong supplementation principle (SSP)"

Basic mereology plus strong supplementation principle is called *extensional mereology(EM)*. Note that one may assume a weaker form of the SSP the so-called weak supplementation principle[5] which is derivable of EM. With the help of EM we get important identity principles and we can show for instance the uniqueness of the mereological sum or intersection.

A6. $\forall xy$ *(eqdim(x,y)* $\rightarrow$ $\exists$*sum(x,y))* "existence of mereological sum"

A7. $\forall xy$ *(sov(x,y)* $\rightarrow$ $\exists$ *intersect(x,y))* "existence of mereological intersection"

A8. $\forall xy$ *(*$\neg$*spart(y,x)* $\wedge$ *eqdim(x,y)* $\rightarrow$ $\exists$*relcompl(x,y))*

"existence of relative complement"

EM plus axioms A6-A8 is called the *classical extensional mereology (CEM)* and every model of CEM is a distributive lattice with relative complements [Bir 1967], [Graet 1998].

4.4.2 Atomicity and Embedding Postulations

Furthermore extensions of CEM are dealing with questions of atomicity respective the existence of a least or greatest element. We assume that space regions, surface regions or line regions are atomless. A point is per definition D26 an atom. We deny the existence of a least element (a bare entity) because if we do this we have to conclude that the intersection of two disjoint spatial entities is not empty. The axiom A11 claims that there is no greatest space region or topoid and furthermore that all lower-dimensional entities

[5] compare [Her 2007] p.4

may be embedded in a topoid.

A9. $\forall x$ *(¬0D(x)* $\rightarrow$ $\exists y$ *sppart(y,x))* "no atomic topoids, surfaces or lines"

A10. $\neg\exists x\forall y$ *(spart(x,y)* $\vee$ *hypp(x,y))* "no least element"

A11. $\forall x\exists y$ *(Top(y)* $\wedge$ *(sppart(x,y)* $\vee$ *hypp(x,y)))* "embedding of all space entities"

4.4.3 Co-Domain of Mereological Functions

The mereological sum, intersection and relative complement is defined for all spatial entities under certain assumptions (compare axioms A6-A8). Because of the quadripartite universe of discourse we have to declare the co-domain of these functions. We assume that they apply to the same class of spatial entities like their arguments are. That means for example that the mereological sum of two surface regions is a surface region. Note that the property ordinariness is not invariant for the mereological sum(compare subsection 4.3.3).

A12. $\forall xy$ *($\exists$sum(x,y)* $\rightarrow$ *eqdim(sum(x,y),x))* "co-domain of mereological sum"

A13. $\forall xy$ *($\exists$intersect(x,y)* $\rightarrow$ *eqdim(intersect(x,y),x))*

"co-domain of mereological intersection"

A14. $\forall xy$ *($\exists$relcompl(x,y)* $\rightarrow$ *eqdim(relcompl(x,y),x))*

"co-domain of mereological relative complement"

4.4.4 Spatial Boundaries and Space Regions

A15. $\forall xy$ *(sb(x,y)* $\rightarrow$ *(2DB(x)* $\wedge$ *SReg(y))* $\vee$ *(1DB(x)* $\wedge$ *2DB(y))* $\vee$ *(0DB(x)* $\wedge$ *1DB(y))*

"range restriction"

A16. $\forall xy$ *(sb(x,y)* $\rightarrow$ $\forall u$ *(sppart(u,x)* $\rightarrow$ *sb(u,y))* "parts of boundaries are boundaries"

Disjointness of Spatial Entities

The spatial universe of discourse is a quadripartite universe and every spatial entity belongs exactly to one class.

A17. $\forall x\ (SB(x) \leftrightarrow \neg SReg(x))$ "two disjoint classes of spatial entities"

A18. $\neg\exists x\ ((2DB(x) \wedge 1DB(x)) \vee (2DB(x) \wedge 0DB(x)) \vee (1DB(x) \wedge 0DB(x))$

"three disjoint classes of spatial boundaries"

Existence of Spatial Entities

In order to avoid a trivial theory (empty universe of discourse) we have to assume at least that the class of one-dimensional entities is not empty. By using the axioms of dependency (A25-A27) one may derive that the other three classes of spatial entities are not empty too. Furthermore space regions are the only class of spatial entities with an assured existence of a belonging spatial boundary[6]. For surfaces and lines we will claim the existence of hyper parts.

A19. $\exists x\ (0D(x))$ "existence of a point"

A20. $\forall x\ (SReg(x) \rightarrow \exists y\ sb(y,x))$ "existence of boundaries"

A21. $\forall x\ (2DB(x) \rightarrow \exists y\ 1dhypp(y,x))$ "existence of hyper parts"

A22. $\forall x\ (1DB(x) \rightarrow \exists y\ 0dhypp(y,x))$ "existence of hyper parts"

Maximal Boundary and Touching Areas

The axiom A23 postulates the existence of a maximal boundary for all spatial entities if they have a spatial boundary. With the axiom A20 one may easily derive that a space region always has to have a maximal boundary. The axiom A24 can only be claimed for two-dimensional touching areas because of the possibility of extraordinariness of lower-dimensional touching areas (compare figure 4.13).

[6] The two-dimensional sphere S^2 is an example of a boundaryless surface.

Note that we will prove the uniqueness of "maximal boundary" and "maximal two-dimensional touching area" in T8 and T9 . That is why we can formulate the axioms in the following way.

A23. $\forall xy$ *(sb(y,x)* $\rightarrow \exists$*MaxB(x))* "existence of maximal boundary"

A24. $\forall xyz$ *(2dtoucharea(x,y,z)* $\rightarrow \exists$*Max2DTouchArea(y,z))*

"existence of maximal two-dimensional touching area"

Dependency of Spatial Boundaries

According to the 1. Brentanian Thesis spatial boundaries cannot exist alone. Note that this postulation does not hold for extraordinary spatial surfaces. Imagine therfor a 2-crosssurface (see figure 4.4). This "double-surface" cannot be a spatial boundary (only a two-dimensional hyper part) of a single space region because of the non-commutativity of the spatial boundary function and the mereological sum (see subsection 4.5.3).

A25. $\forall x$ *(2DB(x)* $\wedge$ *Ord(x)* $\rightarrow \exists y$ *(SReg(y)* $\wedge$ *sb(x,y)))*

"surface regions depend on space regions"

A26. $\forall x$ *(1DB(x)* $\rightarrow \exists y$ *(2DB(y)* $\wedge$ *sb(x,y)))* "line regions depend on surface regions"

A27. $\forall x$ *(0DB(x)* $\rightarrow \exists y$ *(1DB(y)* $\wedge$ *sb(x,y)))* "point regions depend on line regions"

4.4.5 Spatial Coincidence

Two spatial boundaries are coincident if and only if they are co-located, that means they are at the same place (see figure 4.4). We want to understand this relation as an equivalence relation on every class of spatial boundaries. That means the Brentanoraum $\mathbf{B}^3$ is divided into equivalence classes by the coincidence relation.

A28. $\forall xy$ *(scoinc(x,y)* $\rightarrow$ *(sb(x)* $\wedge$ *sb(y))* $\wedge$ *eqdim(x,y))* "range restriction"

A29. $\forall x$ *(sb(x)* $\rightarrow$ *scoinc(x,x))* "reflexivity of coincidence"

A30. $\forall xy$ *(scoinc(x,y)* $\rightarrow$ *scoinc(y,x))* "symmetry of coincidence"

A31. $\forall xyz$ *(scoinc(x,y)* $\wedge$ *scoinc(y,z)* $\rightarrow$ *scoinc(x,z))* "transitivity of coincidence"

4.4.6 Interrelations of Spatial or Hyper Parts and Spatial Coincidence

The coincidence relation implies a notion of equal size and form. That is why there are interdependencies between the spatial part and spatial coincidence relation. Here we want to give six axioms which underline the understanding of these relations.

A32. $\forall xx\text{‘}y$ *(spart(x‘,x)* $\wedge$ *scoinc(x,y)* $\rightarrow \exists$ *y‘ (spart(y‘,y)* $\wedge$ *scoinc(y‘,x‘)))*

"existence of coincident spatial parts"

A33. $\forall xx\text{‘}y$ *(hypp(x‘,x)* $\wedge$ *scoinc(x,y)* $\rightarrow \exists$ *y‘ (hypp(y‘,y)* $\wedge$ *scoinc(y‘,x‘)))*

"existence of coincident hyper parts"

A34. $\forall xx\text{‘}yy\text{‘}$ *(spart(x‘,x)* $\wedge$ *spart(y‘,y)* $\wedge$ *scoinc(x‘,y)* $\wedge$ *scoinc(y‘,x)* $\rightarrow$ *scoinc(x,y))*

"condition for spatial coincidence"

A35. $\forall xx\text{‘}$ *(spart(x‘,x)* $\wedge$ *scoinc(x‘,x)* $\rightarrow$ *x‘=x)*

"condition for equality"

A36. $\forall xy$ *(sov(x,y)* $\wedge$ *2DB(x)* $\wedge$ *2DB(y)* $\wedge$ *scoinc(x,y)* $\rightarrow$ *x=y)*

"condition for equality"

Note that it would be false to postulate the axiom A36 for lines and points. Consider therefor an extraordinary point region which consists of three different coincident ordinary points x, y, z. Compose now the mereological sum of x, y and y, z. These sums are coincident and of course they overlap but they are not equal. Surfaces depend on space region and in theorem T29 we will show that there are not three different ordinary and coincident surfaces. We will show this theorem without the axiom A36.

A37. $\forall xx\text{‘}yy\text{‘}$ *(tangpart(x,y)* $\wedge$ *sb(x‘,x)* $\wedge$ *sb(y‘,y)* $\wedge$ *scoinc(x‘,y‘)*$\rightarrow$ *spart(x´,MaxB(y)))*

"there are no new boundaries"

Note that it would be false to postulate that x‘=y‘ because the maximal boundary of y has not necessarily to be ordinary. A simple counter-example is a lying solid rubber sleeve which is cut through vertical at one position (see figure 4.5). Let y be the whole occupied topoid of the rubber sleeve. Imagine now that the rubber sleeve is cut through vertical a second time (on another position). Let x be the occupied topoid of one half, hence it is a tangential part and furthermore it is possible that both have different spatial boundaries which are coincident, namely two non-overlapping coincident parts of the "double-surface".

4.4.7 Axioms about Ordinariness

A38. $\forall x\ (\neg SReg(x) \wedge Ord(x) \rightarrow \exists y\ (sb(x,y) \wedge Ord(MaxB(y))))$

"ordinary spatial boundaries depend on space entities with ordinary boundaries"

A39. $\forall xy\ (scoinc(x,y) \rightarrow (Ord(x) \wedge Ord(y)) \vee (ExOrd(x) \wedge ExOrd(y))$

"ordinary restriction"

A40. $\forall x\ (ExOrd(x) \rightarrow \exists x`x``\ spart(x`,x) \wedge spart(x``,x) \wedge \neg sov(x`,x``) \wedge scoinc(x`,x``) \wedge Ord(x`) \wedge Ord(x``))$

"existence of ordinary spatial parts"

The following axiom claims that if there is a n-fold non-overlapping division of x and furthermore x and y are coincident, than there is a n-fold non-overlapping division of y with pairwise coincident "building blocks". Note that A41 is an axiom schemata[7].

With the help of this axiom we will derive important theorems about cross-entities (see section 4.5.7).

A41. $\forall xx_1 \ldots x_n y\ (scoinc(x,y) \wedge x{=}sum(x_1,\ldots,x_n) \wedge (\bigwedge_{1 \le i<j \le n} \neg sov(x_i, x_j)) \rightarrow \exists y_1 \ldots y_n\ (y{=}sum(y_1,\ldots,y_n) \wedge (\bigwedge_{1 \le i<j \le n} \neg sov(y_i,y_j)) \wedge (\bigwedge_{i=1}^{n} scoinc(x_i,y_i))))$

"equal 'cardinality´ of coincident entities"

[7]Maybe it is possible to derive this axiom if we postulate a weaker form of A41, namely for a twofold division of x.

4.4.8 (Non-)Overlapping Parts

The following two axioms are important to show that there are no three different coincident surfaces (compare theorem T30). The axiom A42 seems to be very constructed. We want to give an example to make clear that this axiom is adequate to our spatial perception.

Imagine that you do a handstand on the ground (see illustration below). Question: Is it possible now that another non-overlapping object is in contact with your palms? No! That is exactly what axiom A42 claims.

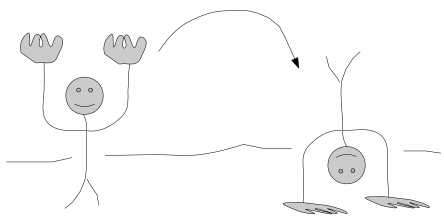


Figure 4.21: Handstand Illustration

A42. $\forall xx'yy'zz'\ (\neg sov(x,y) \wedge x' \neq y' \wedge 2db(x',x) \wedge 2db(y',y) \wedge 2db(z',z) \wedge scoinc(x',y') \wedge scoinc(x',z') \rightarrow \exists p\ spart(p,z) \wedge (spart(p,x) \vee spart(p,y)) \wedge 2db(z',p))$

"a third space region with a coincident boundary has to overlap"

A43. $\forall xx'yy'\ (sov(x,y) \wedge x' \neq y' \wedge 2db(x',x) \wedge 2db(y',y) \wedge scoinc(x',y') \rightarrow \exists z\ (spart(z,x) \wedge \neg sov(z,y) \wedge 2db(x',z)) \vee (spart(z,y) \wedge \neg sov(z,x) \wedge 2db(y',z))$

"existence of a non-overlapping part"

4.5 Propositions

4.5.1 Identity Principles

In this subsection we will prove some important identity principles which are derivable of *extensional mereology* (compare subsection 4.4.1). The following two theorems show that we can define equality in the theory. The 1. identity principle is analog to the extensional principle of set theory. Two entities are identical if and only if they have the same spatial parts (theorem T1) and if and only if they are parts of the same entities (theorem T2).

T1. $\forall xy\ (\forall z\ (spart(z,x) \leftrightarrow spart(z,y)) \leftrightarrow x{=}y)$ "1. identity principle"

Proof:($\Rightarrow$) assume $\forall$z (spart(z,x) $\leftrightarrow$ spart(z,y)); with A1(reflexivity) follows spart(x,x) $\wedge$ spart(y,y); because of the assumption we get **spart(x,y)**(+) $\wedge$ **spart(y,x)**(*); finally with A2(antisymmetry),(+),(*) we conclude **x=y**
($\Leftarrow$) obvious

T2. $\forall xy\ (\forall z\ (spart(x,z) \leftrightarrow spart(y,z)) \leftrightarrow x{=}y)$ "2. identity principle"

Proof:($\Rightarrow$) assume $\forall$z (spart(x,z) $\leftrightarrow$ spart(y,z)); with A1(reflexivity) follows spart(x,x) $\wedge$ spart(y,y); because of the assumption we get **spart(y,x)**(+) $\wedge$ **spart(x,y)**(*); finally with A2(antisymmetry),(+),(*) we conclude **x=y**
($\Leftarrow$) obvious

T3. $\forall xy\ (\exists z\text{‘}\ (sppart(z\text{‘},x)) \wedge \forall z\ (sppart(z,x) \rightarrow sppart(z,y)) \rightarrow spart(x,y))$

"proper part principle"

If every proper part of x is a proper part of y, then x is a part of y under assumption that x has at least one proper part.

Proof: reduction to the absurd; assume ¬spart(x,y) $\wedge$ $\exists$z‘ sppart(z‘,x)(+) $\wedge$ $\forall$z (sppart(z,x) $\rightarrow$ sppart(z,y)(*); consider now two cases;
1. case: ¬**sov(x,y)**; because of (+) we have $\exists$z‘ sppart(z‘,x) and with (*) follows sppart(z‘,y); by D1(proper part) we derive spart(z‘,x) $\wedge$ spart(z‘,y); with D2(spatial overlap) follows **sov(x,y)**;
2. case: sov(x,y); with A5(SSP) and the assumption ¬spart(x,y) we derive $\exists$x‘ spart(x‘,x) $\wedge$ ¬**sov(x‘,y)**; in case of x‘=x follows **sov(x‘,y)**; therefore x‘$\neq$x and with D1(proper part) we derive sppart(x‘,x); by using (*) we get sppart(x‘,y) and finally with D2(spatial overlap) we conclude **sov(x‘,y)**

With the help of theorem T3 we will prove the following 3. identity principle. Two entities are identical if and only if they have the same proper parts under assumption that at least one entity has proper parts. Without this restriction the 3. identity principle would claim that all entities without proper parts are equal which is false in case of points.

T4. $\forall xy\ (\exists z\text{‘}\ (sppart(z\text{‘},x) \vee (sppart(z\text{‘},y)) \rightarrow (x{=}y \leftrightarrow \forall z(sppart(z,x) \leftrightarrow sppart(z,y))))$

"3. identity principle"

Proof:($\Leftarrow$) assume $\exists$z‘ (sppart(z,x) $\vee$ (sppart(z‘,y))(+) and $\forall$z(sppart(z,x) $\leftrightarrow$ sppart(z,y))(*); by D1(proper part), A1(reflexivity) we get $\neg$sppart(x,x) $\wedge$ $\neg$sppart(y,y); with (*) follows $\neg$**sppart(x,y)** $\wedge$ $\neg$**sppart(y,x)**; with (+), (*), T3(proper part principle) we conclude **spart(x,y)** $\vee$ **spart(y,x)**; that means in both cases we derive **x=y**

($\Rightarrow$) this direction is obvious, because of the assumption that x=y

4.5.2 Uniqueness of Mereological Functions

Standard Mereological Functions

T5. $\forall xx‘x_1...x_n(sum(x_1,...,x_n) = x \wedge$ *sum*$(x_1,...,x_n) = x‘ \rightarrow$ *x=x‘)*

"uniqueness of mereological sum"

Proof: reduction to the absurd; assume x$\neq$x‘; by using T1(identity principle) follows w.l.o.g. $\exists$u spart(u,x) $\wedge$ $\neg$spart(u,x‘)(+); with A5(SSP) and (+) we get $\exists$u‘ spart(u‘,u) $\wedge$ $\neg$sov(u‘,x‘)(*); with spart(u,x) and spart(u‘,u) we conclude spart(u‘,x), therefore sov(u´,x); via D3‘(mereological sum) follows $\bigvee_{i=1}^{n}$**sov(u‘,x$_i$)** ; with (*) and D3‘ we derive $\bigwedge_{i=1}^{n}\neg$**sov(u‘,x$_i$)** and this is a contradiction

T6. $\forall xx‘x_1...x_n(intersect(x_1,...,x_n) = x \wedge$ *intersect*$(x_1,...,x_n) = x‘ \rightarrow$ *x=x‘)*

"uniqueness of mereological intersection"

Proof: reduction to the absurd; assume x$\neq$x‘; by using T1(identity principle) follows w.l.o.g. $\exists$u spart(u,x)(+) $\wedge$ $\neg$spart(u,x‘)(*); with (+) and D4‘(mereological intersection) we get $\bigwedge_{i=1}^{n}$**spart(u,x$_i$)**; on the other side with (*) and D4‘(mereological intersection) $\bigvee_{i=1}^{n}\neg$**spart(u,x$_i$)** and this is a contradiction

T7. $\forall xx‘x_1...x_n(relcompl(x_1,...,x_n) = x \wedge$ *relcompl*$(x_1,...,x_n) = x‘ \rightarrow$ *x=x‘)*

"uniqueness of mereological relative complement"

Proof: reduction to the absurd; assume x$\neq$x‘; by using T1(identity principle) follows w.l.o.g. $\exists$u spart(u,x)(+) $\wedge$ $\neg$spart(u,x‘)(*); with (+) and D5‘(relative complement) we get **spart(u,x$_n$)** $\wedge$ $\bigwedge_{i=1}^{n-1}\neg$**sov(u,x$_i$)**; on the other side with (*) and D5‘(relative complement) $\neg$**spart(u,x$_n$)** $\vee$ $\bigvee_{i=1}^{n-1}$**sov(u,x$_i$)** and this is a contradiction

Maximal Boundary and Touching Areas

T8. $\forall xyz$ *(maxb(y,x)* $\wedge$ *maxb(z,x)* $\rightarrow$ *y=z)*

"uniqueness of maximal boundary"

Proof: assume maxb(y,x) $\wedge$ maxb(z,x); by D10(maximal boundary) follows sb(y,x) $\wedge$ sb(z,x), therefore spart(y,z) $\wedge$ spart(z,y); hence **y=z** by antisymmetry of spatial part

T9. $\forall$*xx'yz (MaxTouchArea(y,z)=x* $\wedge$ *MaxTouchArea(y,z)=x´* $\wedge$ *eqdim(x,x')* $\rightarrow$ *x=x')*

"uniqueness of maximal touching area"

Proof: we have to prove three cases;
1. case: assume Max2DTouchArea(y,z)=x $\wedge$ Max2DTouchArea(y,z)=x'; by D52(maximal two-dimensional touching area) follows 2dtoucharea(x,y,z) $\wedge$ 2dtoucharea(x',y,z), therefore spart(x',x) $\wedge$ spart(x,x'); hence **x=x'** by antisymmetry of spatial part
2. and 3. case (maximal one- and zero-dimensional touching areas) in the same way

4.5.3 Non-Commutativity of Maximal Boundary and Mereological Sum

It is an important observation that the maximal boundary function and the mereological sum are not commutative. Consider therefore two spatial squares x and y placed side by side (touching each other). The mereological sum of the maximal boundary of x and the maximal boundary of y is different to the maximal boundary of the mereological sum of x and y. The right side in the following figure represents MaxB(sum(x,y)) and the left side represents sum(MaxB(x),MaxB(y)).

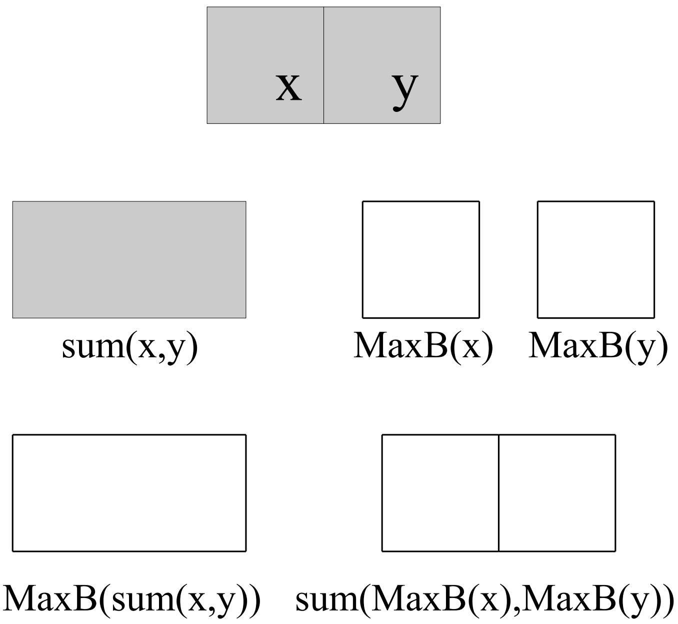


Figure 4.22: Non-Commutativity

4.5.4 Embedding Theorems

In this subsection we will prove embedding theorems for all spatial entities. Note that we have to prove the following result T10 at first.

T10. $\forall xx_1...x_n(sum(x_1,...,x_n) = x\rightarrow \bigwedge_{i=1}^{n} spart(x_i,x))$ "arguments are spatial parts"

Proof: by simple calculation one may prove that ∀xy (spart(x,y) → sov(x,y)(*); now the proof by reduction to the absurd; assume sum(x_1, ...,x_n)=x and w.l.o.g. ¬spart(x_1,x); with A5(SSP) ∃x‘ spart(x´,x_1)(+)∧ ¬**sov(x‘,x)**; with (*), (+) follows sov(x‘,x_1) and with D3‘(mereological sum) we conclude **sov(x‘,x)**

T11. *∀xy (SReg(x) ∧ SReg(y) → ∃z (Top(z) ∧ sppart(x,z) ∧ sppart(y,z)))*

"embedding of space regions"

Proof: assume SReg(x) ∧ SReg(y); by D31(equal dimension) and A6(existence of mereological sum) follows the existence of sum(x,y); because of A12(codomain of mereological sum) the sum is itself a space region; with A11(embedding) we get ∃z **Top(z)** ∧ sppart(sum(x,y),z); because of the theorem T10 above and the transitivity of spatial (proper) part we derive **sppart(x,z)** ∧ **sppart(y,z)**

T12. *∀xy (2DB(x) ∧ 2DB(y) → ∃z (2DB(z) ∧ spart(x,z) ∧ spart(y,z)))*

"embedding of surface regions"

Proof: assume 2DB(x) ∧ 2DB(y); by D31(equal dimension) and A6(existence of mereological sum) follows ∃z sum(x,y)=z; because of A12(codomain of mereological sum) the sum is itself a surface region, thus **2DB(z)**; because of the theorem T10(arguments are spatial parts) we derive **spart(x,z)** ∧ **spart(y,z)**

The following analogical theorems for one- and zero-dimensional entities can be proved in the same way.

T13. *∀xy (1DB(x) ∧ 1DB(y) → ∃z (1DB(z) ∧ spart(x,z) ∧ spart(y,z)))*

"embedding of line regions"

T14. *∀xy (0DB(x) ∧ 0DB(y) → ∃z (0DB(z) ∧ spart(x,z) ∧ spart(y,z)))*

"embedding of point regions"

4.5.5 Spatial Connectedness

In this subsection we want to prove that higher-dimensional connectedness implies lower-dimensional connectedness. Therefore we have to prove at first some preliminary results.

Range of Hyper Parts

T15. *∀xy (2dhypp(x,y) → 2DB(x) ∧ SReg(y))* "range of hyper part relation"

Proof: with D17(two-dimensional hyper part) follows ∃z (spart(z,y)(*) ∧ 2db(x,z); by D12(two-dimensional boundary) we get SReg(z)(+) ∧ sb(x,z) and with A15(range of spatial boundary) follows **2DB(x)**; with (*) and A4(range of spatial part) we have eqdim(z,y) and with (+), D31(equal dimension) follows **SReg(y)**

T16. *∀xy (1dhypp(x,y) → 1DB(x) ∧ (SReg(y) ∨ 2DB(y)))* "range of hyper part relation"

Proof: by D18(one-dimensional hyper part) we have to consider two cases;
1. case: ∃z (spart(z,y)(*) ∧ 1db(x,z)); by D13(one-dimensional boundary) we get 2DB(z)(+) ∧ sb(x,z) and with A15(range of spatial boundary) follows **1DB(x)**; with (*) and A4(range of spatial part) we have eqdim(z,y) and with (+) and D31(equal dimension) follows **2DB(y)**
2. case: ∃z (2dhypp(z,y)(**) ∧ 1db(x,z)(++); with (**) and the theorem above T15 we get immediately **SReg(y)**; by using (++) and D13(one-dimensional boundary) we have 2DB(z) ∧ sb(x,z) and with A15(range of spatial boundary) follows again **1DB(x)**

T17. *∀xy (0dhypp(x,y) → 0DB(x) ∧ (SReg(y) ∨ 2DB(y) ∨ 1DB(y)))*

"range of hyper part relation"

Proof: by D19(one-dimensional hyper part) we have to consider two cases;
1. case: ∃z (spart(z,y)(*) ∧ 0db(x,z); by D14(zero-dimensional boundary) we get 1DB(z)(+) ∧ sb(x,z) and with A15(range of spatial boundary) follows **0DB(x)**; with (*) and A4(range of spatial part) we have eqdim(z,y) and with (+) and D31(equal dimension) follows **1DB(y)**
2. case: ∃z (1dhypp(z,y)(**) ∧ 0db(x,z)(++); with (**) and the theorem above T16 we get **SReg(y)** ∨ **2DB(y)**; by using (++) and D14(zero-dimensional boundary) we have 1DB(z) ∧ sb(x,z) and with A15(range of spatial boundary) follows again **0DB(x)**

Spatial Parts of Hyper Parts

T18. *∀xx'y (2dhypp(x,y) ∧ spart(x',x) → 2dhypp(x´,y))*

"spatial parts of hyper parts are hyper parts"

Proof: assume 2dhypp(x,y) ∧ spart(x',x); with D17(two-dimensional hyper part) follows ∃z (spart(z,y) ∧ 2db(x,z)); with D12(two-dimensional boundary) we get SReg(z)(+) ∧ sb(x,z)(*); by assumption spart(x',x), (*) and A16(parts of boundaries) we derive sb(x',z); by using D12(two-dimensional boundary), (+) and sb(x',z) we get 2db(x',z) and that means **2dhypp(x',y)** because of D17(two-dimensional hyper part)

T19. *∀xx'y (1dhypp(x,y) ∧ spart(x',x) → 1dhypp(x´,y))* "spatial parts of hyper parts are hyper parts"

Proof: assume 1dhypp(x,y) ∧ spart(x‘,x); by D18(one-dimensional hyper part) we have to consider two cases, namely 1. case: ∃z (spart(z,y) ∧ 1db(x,z)) and 2. case: ∃z (2dhypp(z,y) ∧ 1db(x,z)); in both cases it is essential that 1db(x,z); with D13(one-dimensional boundary) we get 2DB(z)(+) ∧ sb(x,z)(*); by assumption spart(x‘,x), (*) and A16(parts of boundaries) we derive sb(x‘,z); by using D13(one-dimensional boundary), (+) and sb(x‘,z) we get 1db(x‘,z) and that means **1dhypp(x‘,y)** for both cases because of D18(one-dimensional hyper part)

T20. *∀xx‘y (0dhypp(x,y) ∧ spart(x‘,x) → 0dhypp(x´,y))*

"spatial parts of hyper parts are hyper parts"

Proof: assume 0dhypp(x,y) ∧ spart(x‘,x); by D19(zero-dimensional hyper part)we have to consider two cases, namely 1. case: ∃z (spart(z,y) ∧ 0db(x,z)) and 2. case: ∃z (1dhypp(z,y) ∧ 0db(x,z)); in both cases it is essential that 0db(x,z); with D14(zero-dimensional boundary) we get 1DB(z)(+) ∧ sb(x,z)(*); by assumption spart(x‘,x), (*) and A16(parts of boundaries) we derive sb(x‘,z); by using D14(zero-dimensional boundary), (+) and sb(x‘,z) we get 0db(x‘,z) and that means **0dhypp(x‘,y)** for both cases because of D19(zero-dimensional hyper part)

Hyper Parts of Hyper Parts

T21. *∀xyz (1dhypp(x,y) ∧ 2dhypp(y,z) → 1dhypp(x,z))* "transitivity of hyper parts"

Proof: assume that 1dhypp(x,y) ∧ 2dhypp(y,z); by theorems T15, T16 follows SReg(z) ∧ 2DB(y) ∧ 1DB(x) ; because of D18(one-dimensional hyper part) y‘ with spart(y‘,y) ∧ 1db(x,y‘) exists(*); by using T18(spatial parts of hyper parts) and 2dhypp(y,z) follows 2dhypp(y‘,z)(+); with (*),(+) and the second part of D18(one-dimensional hyper part) we conclude **1dhypp(x,z)**

T22. *∀xyz (0dhypp(x,y) ∧ 2dhypp(y,z) → 0dhypp(x,z))* "transitivity of hyper parts"

Proof: assume that 0dhypp(x,y) $\wedge$ 2dhypp(y,z); by theorems T15, T17 follows SReg(z) $\wedge$ 2DB(y) $\wedge$ 0DB(x) ; because of D19(zero-dimensional hyper parts) y‘ with 1dhypp(y‘,y) $\wedge$ 0db(x,y‘) exists(*); by using the theorem above T21(transitivity of hyper parts) and 2dhypp(y,z) and 1dhypp(y‘,y) follows 1dhypp(y‘,z)(+); with (*),(+) and the second part of D19(zero-dimensional hyper part) we conclude **0dhypp(x,z)**

T23. $\forall xyz$ *(0dhypp(x,y)* $\wedge$ *1dhypp(y,z)* $\rightarrow$ *1dhypp(x,z))* ”transitivity of hyper parts”

Proof: assume that 0dhypp(x,y) $\wedge$ 1dhypp(y,z); by theorems T16, T17 follows (SReg(z) $\vee$ 2DB(z)) $\wedge$ 1DB(y) $\wedge$ 0DB(x); the dimension of z is irrelevant for the proof; because of D19(zero-dimensional hyper parts) y‘ with spart(y‘,y) $\wedge$ 0db(x,y‘) exists(*); by using T19(parts of hyper parts) and 1dhypp(y,z) follows 1dhypp(y‘,z)(+); with (*),(+) and the second part of D19(zero-dimensional hyper parts) we conclude **0dhypp(x,z)**

Higher and Lower Dimensional Connectedness

T24. $\forall x$ *(2DC(x)* $\rightarrow$ *1DC(x))* ”implication of connectedness”

Proof: reduction to the absurd; assume 2DC(x) $\wedge$ $\neg$1DC(x); by D32(two-dimensional connected) follows SReg(x); with D33(one-dimensional connected) exists a division in y and z with eqdim(y,z) $\wedge$ sum(y,z)=x $\wedge$ $\neg$sov(y,z)(*) and furthermore there are **no one-dimensional hyper parts y´ of y and z´ of z with scoinc(y´,z´)**(+); by using the alternative definition D32‘(two-dimensional connected) we conclude that all divisions which fulfil(*) have to have u, v with 2db(u,y) $\wedge$ 2db(v,z) $\wedge$ scoinc(u,v); note that 2db(u,y) implies 2dhypp(u,y) and analog 2dhypp(v,z), therefore 2DB(u) $\wedge$ 2DB(v) by using T15(range of hyper parts); with A21(existence of hyper parts) follows $\exists$u‘ with 1dhypp(u‘,u) and by using A32(existence of a coincident hyper part) we deduce the $\exists$v‘ with 1dhypp(v‘,v) and **scoinc(u‘,v‘)**; because of T21(hyper parts of hyper parts) follows **1dhypp(u´,y)** $\wedge$ **1dhypp(v‘,z)** and this contradicts (+)

T25. $\forall x$ *(1DC(x)* $\rightarrow$ *0DC(x))* ”implication of connectedness”

Proof: reduction to the absurd; assume 1DC(x) $\wedge$ $\neg$0DC(x); by D33(one-dimensional connected) follows SReg(x) $\vee$ 2DB(x); with D34(zero-dimensional connected) exists a division in y and z with eqdim(y,z) $\wedge$ sum(y,z)=x $\wedge$ $\neg$sov(y,z)(*) and furthermore there are **no zero-dimensional hyper parts y´ of y and z´ of z with scoinc(y´,z´)**(+); by using the alternative definition D33‘(one-dimensional connected) we conclude that all divisions which fulfil(*) have to have u, v with 1dhypp(u,y) $\wedge$ 1dhypp(v,z) $\wedge$ scoinc(u,v); with T16(range of hyper part) follows 1DB(u) $\wedge$ 1DB(v); with A22(existence of hyper parts) follows $\exists$u‘ with 0dhypp(u‘,u) and by using A32(existence of a coincident hyper part) we deduce $\exists$v‘ with 0dhypp(v‘,v) and **scoinc(u‘,v‘)**; because of T23(hyper parts of hyper parts) follows **0dhypp(u‘,y)** $\wedge$ **0dhypp(v‘,z)** and this contradicts (+)

4.5.6 Ordinariness

Preliminary Results

T26. $\forall$*xx‘yy‘ (inpart(x,y)* $\wedge$ *sb(x‘,x)* $\wedge$ *sb(y‘,y)* $\rightarrow$ $\neg$*scoinc(x‘,y‘))*

"there are no coincident boundaries between an entity and their inparts"

Proof: reduction to the absurd; assume inpart(x,y) $\wedge$ sb(x‘,x)(+) $\wedge$ sb(y‘,y)(*) $\wedge$ **scoinc(x‘,y‘)**; with (*) and A23(existence of maximal boundary) follows $\exists$z maxb(z,y); with (*) and D10(maximal boundary) we conclude spart(y‘,z); by D20(hyper part) and (+) we get hypp(x‘,x); with D21(inpart) we conclude **$\neg$scoinc(x‘,y‘)**

T27. $\forall$*xyz (Ord(x)* $\wedge$ *spart(y,x)* $\wedge$ *spart(z,x)* $\wedge$ *scoinc(y,z)* $\rightarrow$ *y=z)*

"condition for equality"

Proof: reduction to the absurd; assume y$\neq$z ; by T1(identity principle) we conclude w.l.o.g. $\exists$y‘ spart(y‘,y) $\wedge$ $\neg$spart(y‘,z); with A5(SSP) we follow $\exists$y“ spart(y“,y‘) $\wedge$ $\neg$**sov(y“,z)**; because of spart(y“,y) $\wedge$ scoinc(y,z) follows with A32(existence of coincident parts) $\exists$z“ spart(z“,z) $\wedge$ scoinc(y“,z“); with D15(ordinary) and Ord(x) $\wedge$ spart(y“,x) $\wedge$ spart(z“,x) follows sov(y“,z“) and therefore **sov(y“,z)** because z“ is part of z

T28. $\forall xyz\ (Ord(MaxB(x)) \wedge sb(y,x) \wedge sb(z,x) \wedge scoinc(y,z) \rightarrow y{=}z)$

"condition for equality"

Proof: with D10(maximal boundary) follows spart(y,MaxB(x)) $\wedge$ spart(z,MaxB(x)); because of the ordinariness of MaxB(x) and scoinc(y,z) and the theorem above T27 we conclude **y=z**

Main Results

T29. $\neg\exists x_1x_2x_3\ (\bigwedge_{i=1,2,3} 2DB(x_i) \wedge Ord(x_i) \wedge \bigwedge_{1\leq i<j\leq 3} x_i \neq x_j \wedge scoinc(x_1,x_2) \wedge scoinc(x_2,x_3))$

"there are no three coincident ordinary surfaces"

Proof: reduction to the absurd; assume $\bigwedge_{1\leq i<j\leq 3} \mathbf{x}_i \neq \mathbf{x}_j$; with A38(ordinary boundaries) follows the existence of y_i with $\bigwedge_{i=1,2,3}$sb(x_i, y_i))$\wedge$ Ord(MaxB(y_i)); consider now two cases
1. case: $\neg$sov(y_1,y_2); with A42(existence of overlapping parts) follows w.l.o.g. $\exists$p spart(p,y_3) $\wedge$ spart(p,y_1)(*) $\wedge$ sb(x_3,p)(+); because of (*),(+),sb(x_1,y_1), scoinc(x_1,x_3) and D21(inpart) we conclude tangpart(p,y_1), hence via A37(no new boundaries) spart(x_3,MaxB(y_1); note that parts of boundaries are boundaries A16, thus sb(x_3,y_1); on account of ordinariness of MaxB(y_1), sb(x_3,y_1), sb(x_1,y_1), scoinc(x_1,x_3) and the theorem above T28 follows $\mathbf{x}_1 = \mathbf{x}_3$ and this contradicts the assumption
2. case: sov(y_1,y_2); with A43(existence of a non-overlapping part) follows $\exists$$y_1$' spart($y_1$',$y_1$) $\wedge$ $\neg$sov(y_1',y_2) $\wedge$ 2db(x_1,y_1'); with A42(existence of overlapping parts) follows w.l.o.g. $\exists$p spart(p,y_3) $\wedge$ spart(p,y_1') $\wedge$ sb(x_3,p)(+); because of transitivity we get spart(p,y_1)(*); by (*), (+), sb(x_1,y_1), scoinc(x_1,x_3) and D21(inpart) we conclude tangpart(p,y_1), hence via A37(no new boundaries) spart(x_3,MaxB(y_1); note that parts of boundaries are boundaries A16, thus sb(x_3,y_1); because of the ordinariness of MaxB(y_1) and sb(x_3,y_1), sb(x_1,y_1), scoinc(x_1,x_3) and T28 follows $\mathbf{x}_1 = \mathbf{x}_3$ and this contradicts the assumption

T30. $\neg\exists x_1x_2x_3\ (\bigwedge_{i=1,2,3} 2DB(x_i) \wedge \bigwedge_{1\leq i<j\leq 3} x_i \neq x_j \wedge scoinc(x_1,x_2) \wedge scoinc(x_2,x_3))$

"there are no three coincident surfaces"

Proof: reduction to the absurd; if x_1, x_2, x_3 are ordinary then nothing is to show, because of the theorem above; hence we have to assume that at least one of them is extraordinary, w.l.o.g. ExOrd(x_1); because of scoinc(x_1,x_2), scoinc(x_2,x_3) and A39(ordinary restriction) we conclude ExOrd(x_2) $\wedge$ ExOrd(x_3); because of A37(condition for equality) we have to assume that $\neg$**sov($\mathbf{x_1}$,$\mathbf{x_2}$)**; by A40(ordinary spatial parts) follows $\exists$ x_1‘ x_1“ spart(x_1‘,x_1) $\wedge$ spart(x_1“,x_1) $\wedge$ x_1‘$\neq$ x_1“ $\wedge$ Ord(x_1‘) $\wedge$ Ord(x_1“) $\wedge$ scoinc(x_1‘,x_1“); with A32(coincident spatial parts) we conclude $\exists x_2$‘ spart(x_2‘,x_2) $\wedge$ scoinc(x_1‘,x_2‘); again with A39(ordinary restriction) we conclude Ord(x_2‘); because of $\bigwedge_{i=1,2,3}$2DB(x_i) and A4(range restriction) we conclude 2DB(x_1‘) $\wedge$ 2DB(x_1“) $\wedge$ 2DB(x_2‘); by T29 we derive x_1‘=x_2‘ $\vee$ x_1“=x_2‘, because there are no three coincident ordinary surfaces; in both cases follows **sov($\mathbf{x_1}$,$\mathbf{x_2}$)**

Note that we did not need the third extraordinary surface for the proof above. That means we have also shown the following stronger theorem.

T31. $\neg\exists x_1 x_2$ *(2DB(x_1) $\wedge$ 2DB(x_2) $\wedge$ ExOrd(x_1) $\wedge$ ExOrd(x_2) $\wedge$ $x_1 \neq x_2$ $\wedge$ scoinc(x_1,x_2))*

"there are no two extraordinary coincident surfaces"

4.5.7 Touching Areas

T32. $\forall xyz$ *(2dtoucharea(x,y,z) $\rightarrow$ Ord(x))*

"two-dimensional touching areas are ordinary"

Proof: reduction to the absurd; assume 2dtoucharea(x,y,z) $\wedge$ **ExOrd(x)**; with D45(two-dimensional touching area) we conclude 2dhypp(x,y); by using T15(range of hyper part) we know **2DB(x)**; again with D45(two-dimensional touching area) follows $\exists$u **u$\neq$x** $\wedge$ 2dhypp(u,z) $\wedge$ **scoinc(u,x)**(+); by T15(range of hyper part) we derive **2DB(u)**; because of (+) and A39(ordinary restriction) we conclude **ExOrd(u)**; the bold marked facts contradict the theorem above

T33. $\forall xyz$ *(toucharea(x,y,z) $\rightarrow$ $\exists x$‘ toucharea(x‘,z,y) $\wedge$ scoinc(x‘,x))*

"existence of a coincident touching area"

Proof: obvious, because of D45-48(touching areas) it is only to show that the external contact of y and z implies the external contact of z and y

T34. *∀xx‘yz (Max2DTouchArea(y,z)=x ∧ 2dtoucharea(x‘,z,y) ∧ scoinc(x‘,x) → x‘=Max2DTouchArea(z,y))*

"condition for maximal two-dimensional touching area"

Proof: reduction to the absurd; assume **x‘≠Max2DTouchArea(z,y)**; because of A24(existence of maximal two-dimensional touching area) we derive ∃u u=Max2DTouchArea(z,y); by D52(maximal two-dimensional touching area) follows spart(x‘,u); because of the theorem above T33 ∃v 2dtoucharea(v,z,y) ∧ scoinc(u,v); by maximality of x follows spart(v,x); because of spart(v,x) ∧ spart(x‘,u) ∧ scoinc(x‘,x) ∧ scoinc(u,v) follows by A34(condition for coincidence) scoinc(x,u) and furthermore with transitivity of spatial coincidence scoinc(x‘,u); hence by A35(condition for equality) u=x‘ and therefore
x‘=Max2DTouchArea(z,y)

Note that theorem T34 cannot be generalized for maximal one- or zero-dimensional touching areas because the existence of a maximal touching area by reason of existence of a touching area can only be claimed for two-dimensional touching areas (compare axiom A24). One- and zero-dimensional touching areas can be extraordinary and this is impossible for two-dimensional touching areas because of the theorem T32.

The following theorem T35 generalizes the conclusion of theorem T33 for maximal two-dimensional touching areas.

T35. *∀xyz (Max2DTouchArea(y,z)=x → ∃x‘ Max2DTouchArea(z,y)=x‘ ∧ scoinc(x‘,x))*

"existence of a maximal two-dimensional coincident touching area"

Proof: because of D52(maximal two-dimensional touching area) and D48(touching area) we get toucharea(x,y,z); with T33 we conclude ∃x‘ toucharea(x‘,z,y) ∧ scoinc(x‘,x); therefore 2dtoucharea(x‘,z,y) by equal dimension of spatial coincidence and finally with T34 **x‘=Max2DTouchArea(z,y)**

4.5.8 Cross-Entities

Preliminary Results for Crosspoints

T36. $\forall x x_1 ... x_{n+1}(x = sum(x_1, ..., x_n) \wedge (\bigwedge_{i=1}^{n+1} 0D(x_i)) \wedge spart(x_{n+1}, x) \rightarrow \bigvee_{i=1}^{n} x_i = x_{n+1})$

"condition for equality"

Proof: by spart(x_{n+1},x) we get sov(x_{n+1},x); because of x=sum($x_1, ..., x_n$) and D3‘(mereological sum) we derive $\bigvee_{i=1}^{n}$sov(x_i,x_{n+1})(+); given that $x_1, ..., x_{n+1}$ are points(no proper parts) follows with (+) $\bigvee_{i=1}^{n}\mathbf{x}_i\mathbf{=x}_{n+1}$

T37. $\forall x x_1 ... x_n x_1‘ ... x_m‘(x = sum(x_1, ..., x_n) = sum(x_1‘, ..., x_m‘) \wedge (\bigwedge_{i=1}^{n} 0D(x_i)) \wedge (\bigwedge_{i=1}^{m} 0D(x_i‘)) \rightarrow \bigwedge_{i=1}^{n}(\bigvee_{j=1}^{m} x_i = x_j‘))$

"condition for equality"

Proof: reduction to the absurd; assume $\neg(\bigwedge_{i=1}^{n}(\bigvee_{j=1}^{m}x_i$ =x$_j$‘)), that means w.l.o.g. $\bigwedge_{j=1}^{m}\mathbf{x}_1 \neq \mathbf{x}_j$**‘**; because of x=sum($x_1, ..., x_n$) and T10(arguments are spatial parts) we get spart(x_1,x)(+); by x=sum(x_1‘,...,x_m‘) ∧0D(x_1) ∧ $\bigwedge_{i=1}^{m}$0D(x_i‘) and (+) follows with T36 $\bigvee_{i=1}^{m}\mathbf{x}_1\mathbf{=x}_i$‘

T38. $\forall x x_1 ... x_n x_1‘ ... x_m‘(x = sum(x_1, ..., x_n) \wedge (\bigwedge_{i=1}^{n} 0D(x_i)) \wedge (\bigwedge_{1 \leq i<j \leq n} x_i \neq x_j) \wedge x = sum(x_1‘, ..., x_m‘) \wedge (\bigwedge_{i=1}^{m} 0D(x_i‘)) \wedge (\bigwedge_{1 \leq i<j \leq m} x_i‘ \neq x_j‘) \rightarrow equ(x_1, ..., x_n, x_1‘, ... x_m‘))$

"condition for equality"

Proof: the pairwise inequality of x_1,...,x_n and x_1‘,...,x_m‘ is given by premise; because of T37(condition for equality) is every x_i a x_j‘ and vice versa; furthermore we have to guarantee that n=m; if we assume n≠m, that means w.l.o.g n<m follows the existence of x_i‘, x_j‘ and x_k with x_i‘≠x_j‘, but x_i‘=x_k ∧ x_j‘=x_k and this is a contradiction; altogether we have **equ($\mathbf{x}_1$,...,$\mathbf{x}_n$,$\mathbf{x}_1$‘,...$\mathbf{x}_m$‘)**

T39. $\forall x x_1 ... x_n x_1‘ ... x_m‘(x = sum(x_1, ..., x_n) \wedge (\bigwedge_{i=1}^{n} 0D(x_i)) \wedge (\bigwedge_{1 \leq i<j \leq n} x_i \neq x_j \wedge scoinc(x_i, x_j)) \wedge x = sum(x_1‘, ..., x_m‘) \wedge (\bigwedge_{i=1}^{m} 0D(x_i‘)) \wedge (\bigwedge_{1 \leq i<j \leq m} x_i‘ \neq x_j‘ \wedge scoinc(x_i‘, x_j‘)) \rightarrow equ(x_1, ..., x_n, x_1‘, ... x_m‘))$

"a n-crosspoint is no m-crosspoint (m≠n)"

Proof: directly with T38(condition for equality), because it is only an additional premise

The following notation of theorem T39 is in second order logic. This notation is more understandable then the first order version.

T39‘. $\forall x,n,m$ *(n-crosspoint(x)* $\wedge$ *m-crosspoint(x)* $\wedge$ $n,m \in \boldsymbol{N} \rightarrow n=m)$

"a n-crosspoint is no m-crosspoint(m≠n)"

T40. $\forall x$ *(0D(x)* $\rightarrow$ *Ord(x))*

"points are ordinary"

Proof: reduction to the absurd; assume 0D(x) ∧ ExOrd(x); by using D16(extraordinary) we get ∃x‘x“ (spart(x‘,x) ∧ spart(x“,x) ∧ ¬sov(x‘,x“)); if we furthermore assume that x‘=x ∧ x“=x we have sov(x‘,x“), so w.l.o.g. x‘≠x and this yields with spart(x‘,x) to sppart(x‘,x) by D1(spatial proper part) and this contradicts D44(point), because points have no spatial proper parts

T41. $\forall xy$ *(scoinc(x,y)* $\wedge$ *0D(x)* $\rightarrow$ *0D(y))*

"points may coincident with points"

Proof: reduction to the absurd; assume scoinc(x,y) ∧ 0D(x) ∧ ¬0D(y); with scoinc(x,y) ∧ 0D(x), A28(range of coincidence) and D31(equal dimension) we get 0DB(y)(+); because of D44(point) and (+) follows the **existence of y‘ with sppart(y‘,y)**(*); by using A32(existence of coincident spatial parts),(*) and scoinc(x,y) we get ∃x‘ spart(x‘,x) ∧ scoinc(x‘,y‘); by using that x is a point(no proper parts) we get x‘=x; by transitivity of spatial coincidence follows scoinc(y‘,y); finally with A35(condition for equality) follows **y‘=y** and this contradicts (*)

T42. $\forall x x_1 ... x_n y$ *(scoinc(x,y)* $\wedge$ *x=sum*$(x_1, ..., x_n)$ $\wedge$ $(\bigwedge_{i=1}^{n} \mathit{0D}(x_i))$ $\wedge$ $(\bigwedge_{1 \leq i<j \leq n} x_i \neq x_j) \rightarrow$ $\exists y_1 ... y_n$ *(y=sum*$(y_1, ..., y_n)$ $\wedge$ $(\bigwedge_{i=1}^{n} \mathit{0D}(y_i) \wedge \mathit{scoinc}(x_i, y_i))$ $\wedge$ $(\bigwedge_{1 \leq i<j \leq m} y_i \neq y_j$ *)))*

"existence of a coincident division"

Proof: because of 0D(x_i) and the pairwise inequality of them follows $\bigwedge_{1 \leq i<j \leq n}$ ¬sov(x_i,x_j)(+); with the premise,(+) and A41(equal cardinality) we get $\exists \mathbf{y}_1 ... \mathbf{y}_n$ **(y=sum($\mathbf{y}_1$,...,$\mathbf{y}_n$)** $\wedge \bigwedge_{1 \leq i<j \leq n}$¬sov($y_i$,$y_j$)(*) $\wedge$ $\bigwedge_{i=1}^{n}$ **scoinc($\mathbf{x}_i$,$\mathbf{y}_i$)**); by using (*) we get $\bigwedge_{1 \leq i<j \leq n} \mathbf{y}_i \neq \mathbf{y}_j$; because of $\bigwedge_{i=1}^{n}$ scoinc(x_i,y_i) ∧ 0D(x_i) and T41(points coincident with points) follows $\bigwedge_{i=1}^{n}$ **0D($\mathbf{y}_i$)**

Main Result for Crosspoints

The following theorem T43 or T43‘ declares that if x is a n-crosspoint plus coincidence of x and y, then y is a n-crosspoint, too. By using the conclusion of theorem T39 that a n-crosspoint is no m-crosspoint we can formulate the main result namely that only crosspoints with the same cardinality[8] in the sense of n=m may coincide (theorem T44).

T43. $\forall x x_1 ... x_n y$ *(scoinc(x,y)* $\wedge$ *x=sum*$(x_1, ..., x_n)$ $\wedge$ $(\bigwedge_{i=1}^{n} 0D(x_i))$ $\wedge$ $(\bigwedge_{1\leq i<j\leq n} x_i \neq x_j \wedge scoinc(x_i, x_j)) \rightarrow \exists y_1 ... y_n$ *(y=sum*$(y_1, ..., y_n)$ $\wedge$ $(\bigwedge_{i=1}^{n} 0D(y_i))$ $\wedge$ $(\bigwedge_{1\leq i<j\leq m} y_i \neq y_j \wedge scoinc(y_i, y_j))))$

"n-crosspoints coincident with n-crosspoint"

> *Proof:* because of T42(existence of a coincident division)we have only to show that $\bigwedge_{1\leq i<j\leq n}$scoinc($y_i$,$y_j$); this is easy to show, because of $\bigwedge_{1\leq i<j\leq n}$scoinc($x_i$,$x_j$) (premise) and $\bigwedge_{i=1}^{n}$scoinc(x_i,y_i)(conclusion of T42) and using of transitivity of spatial coincidence

The following notation of theorem T43 in second order logic provides a better notion of the result.

T43‘. $\forall$*x,y,n (scoinc(x,y)* $\wedge$ *n-crosspoint(x)* $\wedge$ $n\in\boldsymbol{N} \rightarrow$ *n-crosspoint(y))*

"n-crosspoints coincident with n-crosspoint"

T44. $\forall x x_1 ... x_n y y_1 ... y_m (scoinc(x, y) \wedge x{=}sum(x_1, ..., x_n) \wedge y{=}sum(y_1, ..., y_m) \wedge (\bigwedge_{i=1}^{n} 0D(x_i)) \wedge (\bigwedge_{1\leq i<j\leq n} x_i \neq x_j \wedge scoinc(x_i, x_j)) \wedge (\bigwedge_{i=1}^{m} 0D(y_i)) \wedge (\bigwedge_{1\leq i<j\leq m} y_i \neq y_j \wedge scoinc(y_i, y_j)) \rightarrow \exists y_1‘ ... y_m‘(equ(y_1, ..., y_n, y_1‘, ... y_m‘)))$

"n-crosspoints **only** coincident with n-crosspoint"

We will prove this theorem in the second order version, but note that it is possible to do this in first order too.

T44‘. $\forall$*x,y,n,m (scoinc(x,y)* $\wedge$ *n-crosspoint(x)* $\wedge$ *m-crosspoint(y)* $\wedge$ *n,m*$\in\boldsymbol{N} \rightarrow$ *n=m)*

"n-crosspoints **only** coincident with n-crosspoint"

[8] Note that it is impossible to define equal cardinality in first order logic in general. In our special case we can show the pairwise equality of the "building blocks" of an alternative division (see theorem T39) and this implies obviously equal cardinality.

Proof: reduction to the absurd; assume scoinc(x,y) $\wedge$ n-crosspoint(x) $\wedge$ m-crosspoint(y) $\wedge$ **n≠m**; with T43‘(n-crosspoints coincident with n-crosspoint) we get n-crosspoint(y) and finally with T39‘(n-crosspoint is no m-crosspoint) follows **n=m**

Preliminary Results for Crosslines

The next both theorems T45 and T46 express that a n-crossline is no m-crossline (with m<n respective n≠m). We will give more comprehensible versions in T45‘ and T46‘.

T45. $\forall x x_1 ... x_n$ $(x{=}sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^{n} 1D(x_i) \wedge Ord(x_i)) \wedge (\bigwedge_{1\leq i<j\leq n} \neg\ sov(x_i,x_j) \wedge scoinc(x_i,x_j)) \wedge \bigwedge_{i=1}^{n-1}(\neg\exists x_1\text{‘}...x_i\text{‘}(x{=}sum(x_1\text{‘},...,x_i\text{‘}) \wedge (\bigwedge_{k=1}^{i} 1D(x_k\text{‘}) \wedge\ Ord(x_k\text{‘})) \wedge$ $(\bigwedge_{1\leq k<l\leq i} \neg sov(x_k\text{‘},x_l\text{‘}) \wedge scoinc(x_k\text{‘},x_l\text{‘})))) \rightarrow (\bigwedge_{j=1}^{n-1} \neg\exists y_1...y_j((x{=}sum(y_1,...,y_j) \wedge$ $(\bigwedge_{k=1}^{j} 1D(y_k) \wedge\ Ord(y_k)) \wedge (\bigwedge_{1\leq k<l\leq j} \neg sov(y_k,y_l) \wedge scoinc(y_k,y_l))) \wedge \bigwedge_{i=1}^{j-1}(\neg\exists y_1\text{‘}...y_i\text{‘}$ $(x{=}sum(y_1\text{‘},...,y_i\text{‘}) \wedge (\bigwedge_{k=1}^{i} 1D(y_k\text{‘}) \wedge\ Ord(y_k\text{‘})) \wedge (\bigwedge_{1\leq k<l\leq i} \neg sov(y_k\text{‘},y_l\text{‘}) \wedge$ $scoinc(y_k\text{‘},y_l\text{‘})))))))$

"a n-crossline is no 1-,...,(n-1)-crossline"

Proof: obvious, because the assumption of the negation of the conclusion contradicts the premise

T45‘. $\forall x,n$ (n-crossline(x) $\wedge$ n$\in$**N** $\rightarrow \neg\exists$m (m$\in$**N** $\wedge$ m<n $\wedge$ m-crossline(x)))

"a n-crossline is no 1-,...,(n-1)-crossline"

T46. $\forall x x_1 ... x_n$ $(x{=}sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^{n} 1D(x_i) \wedge Ord(x_i)) \wedge (\bigwedge_{1\leq i<j\leq n} \neg\ sov(x_i,x_j) \wedge scoinc(x_i,x_j)) \wedge \bigwedge_{i=1}^{n-1}(\neg\exists x_1\text{‘}...x_i\text{‘}(x{=}sum(x_1\text{‘},...,x_i\text{‘}) \wedge (\bigwedge_{k=1}^{i} 1D(x_k\text{‘}) \wedge\ Ord(x_k\text{‘})) \wedge$ $(\bigwedge_{1\leq k<l\leq i} \neg sov(x_k\text{‘},x_l\text{‘}) \wedge scoinc(x_k\text{‘},x_l\text{‘})))) \rightarrow (\bigwedge_{j\in \boldsymbol{N},j\neq n} \neg\exists y_1...y_j((x{=}sum(y_1,...,y_j) \wedge$ $(\bigwedge_{k=1}^{j} 1D(y_k) \wedge\ Ord(y_k)) \wedge (\bigwedge_{1\leq k<l\leq j} \neg sov(y_k,y_l) \wedge scoinc(y_k,y_l))) \wedge \bigwedge_{i=1}^{j-1}(\neg\exists y_1\text{‘}...y_i\text{‘}$ $(x{=}sum(y_1\text{‘},...,y_i\text{‘}) \wedge (\bigwedge_{k=1}^{i} 1D(y_k\text{‘}) \wedge\ Ord(y_k\text{‘})) \wedge (\bigwedge_{1\leq k<l\leq i} \neg sov(y_k\text{‘},y_l\text{‘}) \wedge$ $scoinc(y_k\text{‘},y_l\text{‘})))))))$

"a n-crossline is no m-crossline(n≠m)"

Proof: reduction to the absurd; assume the existence of a second non-overlapping division of x with j≠n; both cases(j<n and n<j) lead to a contradiction by using the result of T45(n-crossline is no 1-,...,(n-1)-crossline)

T46‘. $\forall x,n$ *(n-crossline(x)* $\wedge$ $n \in \boldsymbol{N} \rightarrow \neg\exists m$ $(m \in \boldsymbol{N} \wedge n \neq m \wedge$ *m-crossline(x)))*

"a n-crossline is no m-crossline($n \neq m$)"

T47. $\forall xy$ *(scoinc(x,y)* $\wedge$ *1D(x)* $\wedge$ *Ord(x)* $\rightarrow$ *1D(y)* $\wedge$ *Ord(y))*

"ordinary lines may coincident with ordinary lines"

Proof: reduction to the absurd; assume scoinc(x,y) $\wedge$ 1D(x) $\wedge$ Ord(x) $\wedge$ $\neg$(1D(y) $\wedge$ Ord(y)); the ordinariness of y is given by Ord(x), scoinc(x,y) and A39(ordinary restriction); by assumption follows $\neg$1D(y)(+); with scoinc(x,y) $\wedge$ 1D(x), A28(range of coincidence) and D31(equal dimension) we get 1DB(y)(*); by using (+), (*) and D43(line) follows $\neg$0DC(y), that means with D34(zero-dimensional connected) there is a division in y_1 and y_2 with (eqdim(y_1,y_2) $\wedge$ sum(y_1,y_2)=y $\wedge$ $\neg$sov(y_1,y_2))(++) and furthermore **there are no zero-dimensional hyper parts $\mathbf{y_1}$‘ of $\mathbf{y_1}$ and $\mathbf{y_2}$‘ of $\mathbf{y_2}$ with scoinc($\mathbf{y_1}$‘,$\mathbf{y_2}$‘)**; with (++) and A41(equal cardinality) we get the existence of a division in x_1 and x_2 with (eqdim(x_1,x_2) $\wedge$ sum(x_1,x_2)=x $\wedge$ $\neg$sov(x_1,x_2))(**) $\wedge$ scoinc(x_1,y_1) $\wedge$ scoinc(x_2,y_2); x is zero-dimensional connected, because of 1D(x)(see definition D43); by using (**) and D34‘(zero-dimensional connected) we conclude the existence of zero-dimensional hyper parts x_1‘ of x_1 and x_2‘ of x_2 with scoinc(x_1‘,x_2‘); with A33(existence of coincident hyper parts), scoinc(x_1,y_1) and hypp(x_1‘,x_1) we get the **existence of a zero-dimensional hyper part $\mathbf{y_1}$‘ of $\mathbf{y_1}$** with scoinc(y_1‘,x_1‘) **and** analogous the **existence of a zero-dimensional hyper part $\mathbf{y_2}$‘ of $\mathbf{y_2}$** with scoinc(y_2‘,x_2‘); finally with the transitivity of spatial coincidence and scoinc(y_1‘,x_1‘), scoinc(y_2‘,x_2‘) and scoinc(x_1‘,x_2‘) we get **scoinc($\mathbf{y_1}$‘,$\mathbf{y_2}$‘)** and this is a contradiction

T48. $\forall x x_1 ... x_n y$ *(scoinc(x,y)* $\wedge$ *x=sum*$(x_1, ..., x_n)$ $\wedge$ $(\bigwedge_{i=1}^{n}$ *1D*(x_i) $\wedge$ *Ord*$(x_i))$ $\wedge$ $(\bigwedge_{1 \leq i < j \leq n}$ $\neg$ *sov*$(x_i, x_j))\rightarrow$ $\exists y_1 ... y_n$ *(y=sum*$(y_1, ..., y_n)$*)* $\wedge$ $(\bigwedge_{i=1}^{n}$ *1D*$(y_i)\wedge$ *Ord*$(y_i)\wedge$ *scoinc*$(x_i, y_i))$ $\wedge$ $(\bigwedge_{1 \leq i < j \leq m} \neg sov(y_i, y_j))))$

"existence of coincident building blocks"

Proof: by using x=sum(x_1,...,x_n) $\wedge \bigwedge_{1 \leq i < j \leq n}\neg$sov($x_i$,$x_j$) $\wedge$ scoinc(x,y) and A41(equal cardinality) we get **$\exists \mathbf{y_1} ... \mathbf{y_n}$ (y=sum($\mathbf{y_1}$,...,$\mathbf{y_n}$)** $\wedge \bigwedge_{1 \leq i < j \leq n}$**$\neg$sov($\mathbf{y_i}$,$\mathbf{y_j}$)** $\wedge$ $\bigwedge_{i=1}^{n}$ **scoinc($\mathbf{x_i}$,$\mathbf{y_i}$)**; with 1D(x_i) $\wedge$ Ord(x_i)$\wedge$ scoinc(x_i,y_i) for all i with $1 \leq i \leq n$ and T47(ordinary lines coincident with ordinary lines) follows $\bigwedge_{i=1}^{n}$ **1D($\mathbf{y_i}$)** $\wedge$ **Ord($\mathbf{y_i}$)**

Starting from now we will call the conclusion of T49 a "n-division of y". With the help of this agreement we can formulate the following theorems in a more comprehensible way.

T49. $\forall xx_1...x_n y$ *(scoinc(x,y)* $\wedge$ *x=sum*$(x_1,...,x_n)$ $\wedge$ $(\bigwedge_{i=1}^{n}$*1D*(x_i) $\wedge$ *Ord*$(x_i))$ $\wedge$ $(\bigwedge_{1\leq i<j\leq n}$ $\neg$ *sov*$(x_i,x_j)\wedge$ *scoinc*$(x_i,x_j))\rightarrow$ $\exists y_1...y_n$ *(y=sum*$(y_1,...,y_n)$ $\wedge$ $(\bigwedge_{i=1}^{n}$*1D*$(y_i)\wedge$ *Ord*$(y_i))$ $\wedge$ $(\bigwedge_{1\leq i<j\leq m}\neg$*sov*$(y_i,y_j)$ $\wedge$ *scoinc*$(y_i,y_j))))$

"n-divisions coincident with n-divisions"

> *Proof:* because of T48(existence of coincident building blocks)we only have to show that $\bigwedge_{1\leq i<j\leq n}$scoinc($y_i$,$y_j$); this is easy to show, because of $\bigwedge_{1\leq i<j\leq n}$scoinc($x_i$,$x_j$) (premise) and $\bigwedge_{i=1}^{n}$scoinc(x_i,y_i)(conclusion of T48) and using of transitivity of spatial coincidence

T49'. $\forall$*x,y,n (scoinc(x,y)* $\wedge$ *n-division(x)* $\wedge$ *n*$\in$ ***N***$\rightarrow$ *n-division(y))*

"n-divisions coincident with n-divisions"

T50. $\forall xx_1...x_n y$ *(scoinc(x,y)* $\wedge$ *x=sum*$(x_1,...,x_n)$ $\wedge$ $(\bigwedge_{i=1}^{n}$*1D*(x_i) $\wedge$ *Ord*$(x_i))$ $\wedge$ $(\bigwedge_{1\leq i<j\leq n}$ $\neg$ *sov*$(x_i,x_j)\wedge$ *scoinc*$(x_i,x_j))\wedge$ $(\bigwedge_{i=1}^{n-1}(\neg\exists x_1`...x_i`$*(x=sum*$(x_1`,...,x_i`)$ $\wedge$ $(\bigwedge_{k=1}^{i}$*1D*$(x_k`)\wedge$ *Ord*$(x_k`))$ $\wedge$ $(\bigwedge_{1\leq k<l\leq i}\neg$*sov*$(x_k`,x_l`)$ $\wedge$*scoinc*$(x_k`,x_l`)))))$ $\rightarrow$ $\exists y_1...y_n$ *(y=sum*$(y_1,...,y_n)$ $\wedge$ $(\bigwedge_{i=1}^{n}$*1D*$(y_i)\wedge$ *Ord*$(y_i))$ $\wedge$ $(\bigwedge_{1\leq i<j\leq m}\neg$*sov*$(y_i,y_j)$ $\wedge$ *scoinc*$(y_i,y_j))))$

"n-crosslines coincident with n-divisions"

> *Proof:* directly with T49(n-divisions coincident with n-divisions) because it is only an additional premise

T50'. $\forall$*x,y,n (scoinc(x,y)* $\wedge$ *n-crossline(x)* $\wedge$ *n*$\in$ ***N***$\rightarrow$ *n-division(y))*

"n-crosslines coincident with n-divisions"

Main Results for Crosslines

T51. $\forall xx_1...x_n y$ *(scoinc(x,y)* $\wedge$ *x=sum*$(x_1,...,x_n)$ $\wedge$ $(\bigwedge_{i=1}^{n}$*1D*(x_i) $\wedge$ *Ord*$(x_i))$ $\wedge$ $(\bigwedge_{1\leq i<j\leq n}\neg$ *sov*$(x_i,x_j)\wedge$*scoinc*$(x_i,x_j))\wedge$ $\bigwedge_{i=1}^{n-1}(\neg\exists x_1`...x_i`$*(x=sum*$(x_1`,...,x_i`)\wedge(\bigwedge_{k=1}^{i}$*1D*$(x_k`)\wedge$ *Ord*$(x_k`))\wedge$ $(\bigwedge_{1\leq k<l\leq i}\neg$*sov*$(x_k`,x_l`)$ $\wedge$*scoinc*$(x_k`,x_l`))))$ $\rightarrow$ $\exists y_1...y_n$*(y=sum*$(y_1,...,y_n)$ $\wedge$ $(\bigwedge_{i=1}^{n}$*1D*(y_i) $\wedge$ *Ord*$(y_i))$ $\wedge$ $(\bigwedge_{1\leq i<j\leq n}\neg$ *sov*(y_i,y_j) $\wedge$ *scoinc*$(y_i,y_j))\wedge$ $\bigwedge_{i=1}^{n-1}(\neg\exists y_1`...y_i`$*(y=sum*$(y_1`,...,y_i`)$ $\wedge$ $(\bigwedge_{k=1}^{i}$*1D*$(y_k`)\wedge$ *Ord*$(y_k`))$ $\wedge$ $(\bigwedge_{1\leq k<l\leq i}\neg$*sov*$(y_k`,y_l`)$ $\wedge$*scoinc*$(y_k`,y_l`))))$

"n-crosslines coincident with n-crosslines"

We will prove the theorems T51 and T52 in their second order version but note that it is possible to do this in first order, too.

T51'. $\forall$*x,y,n (scoinc(x,y)* $\wedge$ *n-crossline(x)* $\wedge$ *n*$\in\boldsymbol{N}\rightarrow$ *n-crossline(y)))*

"n-crosslines coincident with n-crosslines"

Proof: reduction to the absurd; assume scoinc(x,y) $\wedge$ **n-crossline(x)** $\wedge$ $\neg$n-crossline(y)(+); with T50'(n-crosslines coincident with n-divisions) follows n-division(y)(*); because of (+),(*) and D58(n-crossline) we conclude $\exists$j (j$\in$**N** $\wedge$ 1$\leq$j<n $\wedge$ j-division(y)(++)); by using (++), scoinc(x,y) and T49'(n-divisions coincident with n-divisions) we get **j-division(x)** and this contradicts the definition of a n-crossline(D58)

T52. $\forall x x_1 ... x_n y y_1 ... y_m$ $(scoinc(x,y) \wedge x{=}sum(x_1,...,x_n) \wedge (\bigwedge_{i=1}^{n} 1D(x_i) \wedge Ord(x_i)) \wedge (\bigwedge_{1\leq i<j\leq n} \neg\, sov(x_i,x_j) \wedge scoinc(x_i,x_j)) \wedge \bigwedge_{i=1}^{n-1}(\neg\exists x_1`...x_i`(x{=}sum(x_1`,...,x_i`) \wedge (\bigwedge_{k=1}^{i} 1D(x_k`) \wedge Ord(x_k`)) \wedge (\bigwedge_{1\leq k<l\leq i} \neg sov(x_k`,x_l`) \wedge scoinc(x_k`,x_l`)))) \wedge y{=}sum(y_1,...,y_m) \wedge (\bigwedge_{i=1}^{m} 1D(y_i) \wedge Ord(y_i)) \wedge (\bigwedge_{1\leq i<j\leq m} \neg\, sov(y_i,y_j) \wedge scoinc(y_i,y_j)) \wedge \bigwedge_{i=1}^{m-1}(\neg\exists y_1`...y_i`(y{=}sum(y_1`,...,y_i`) \wedge (\bigwedge_{k=1}^{i} 1D(y_k`) \wedge Ord(y_k`)) \wedge (\bigwedge_{1\leq k<l\leq i} \neg sov(y_k`,y_l`) \wedge scoinc(y_k`,y_l`)))) \rightarrow \exists z_1...z_n\ equ(z_1,...,z_n,y_1,...,y_m))$

"n-crosslines **only** coincident with n-crosslines"

T52'. $\forall$*x,y,n,m (scoinc(x,y)* $\wedge$ *n-crossline(x)* $\wedge$ *m-crossline(y)* $\wedge$ *n,m*$\in\boldsymbol{N}\rightarrow$ *n=m)*

"n-crosslines **only** coincident with n-crosslines"

Proof: with scoinc(x,y) $\wedge$ n-crossline(x) and T51'(n-crosslines coincident with n-crosslines) follows n-crossline(y) and finally with T46'(a n-crossline is no m-crossline) we get **n=m**

Preliminary Results for Crosssurfaces

The following results show that there are no n-crosssurfaces with n$\geq$3. That means there are only two kinds of crosssurfaces, namely the extraordinary 2-crosssurface and the ordinary 1-crosssurface (theorem T54). Furthermore we will show that if x is a 2-crosssurface, then there is no other coincident spatial entity y. Remember that lower-dimensional cross-entities may coincident with other cross-entities. The theorem T59' summarizes all results of this section.

T53. $\neg\exists x x_1 x_2 x_3 ... x_n (x = sum(x_1, x_2, x_3, ..., x_n) \wedge (\bigwedge_{i=1}^{n} 2D(x_i) \wedge Ord(x_i)) \wedge (\bigwedge_{1\leq i<j\leq n} \neg sov(x_i, x_j) \wedge scoinc(x_i, x_j))) \wedge (\bigwedge_{i=1}^{n-1} (\neg\exists x_1\text{‘} x_i\text{‘} (x = sum(x_1\text{‘}, x_i\text{‘}) \wedge (\bigwedge_{k=1}^{i} 2D(x_k\text{‘}) \wedge Ord(x_k\text{‘})) \wedge (\bigwedge_{1\leq k<l\leq i} \neg sov(x_k\text{‘}, x_l\text{‘}) \wedge scoinc(x_k\text{‘}, x_l\text{‘})))))$

"there are no n-crosssurfaces(n≥3)"

Proof: the first part of the conjunction contradicts the main result for surface regions T30(there are no three coincident surfaces)

T54. $\forall x x_1 x_2 (x = sum(x_1, x_2) \wedge (\bigwedge_{i=1}^{2} 2D(x_i) \wedge Ord(x_i)) \wedge \neg sov(x_1, x_2) \wedge scoinc(x_1, x_2) \wedge \neg\exists x\text{‘}(x = sum(x\text{‘}) \wedge 2D(x\text{‘}) \wedge Ord(x\text{‘})) \rightarrow \neg\exists y\text{‘}(x = sum(y\text{‘}) \wedge 2D(y\text{‘}) \wedge Ord(y\text{‘}))$

"a 2-crosssurface is no 1-crosssurface"

Proof: obvious, because it is a tautology

T54‘. $\forall x$ *(2-crosssurface(x)* $\rightarrow \neg$ *1-crosssurface(x))*

"a 2-crosssurface is no 1-crosssurface"

T55. $\forall x x_1 x_2 y (x = sum(x_1, x_2) \wedge (\bigwedge_{i=1}^{2} 2D(x_i) \wedge Ord(x_i)) \wedge \neg sov(x_1, x_2) \wedge scoinc(x_1, x_2) \wedge \neg\exists x\text{‘}(x = sum(x\text{‘}) \wedge 2D(x\text{‘}) \wedge Ord(x\text{‘})) \wedge 2D(y) \wedge Ord(y) \rightarrow \neg scoinc(x,y))$

"a 2-crosssurface cannot coincide with a 1-crosssurface"

Proof: obvious, because x is extraordinary and y is ordinary (contradiction to A39 ordinary restriction)

T55‘. $\forall xy$ *(2-crosssurface(x)* $\wedge$ *1-crosssurface(y)* $\rightarrow \neg$*scoinc(x,y))*

"a 2-crosssurface cannot coincide with a 1-crosssurface"

Main Result for Crosssurfaces

T56. $\forall xy\ (scoinc(x,y) \wedge 2D(x) \wedge Ord(x) \rightarrow 2D(y) \wedge Ord(y))$

"ordinary surfaces may coincide with ordinary surfaces"

Proof: reduction to the absurd; assume scoinc(x,y) ∧ 2D(x) ∧ Ord(x) ∧ ¬(2D(y) ∧ Ord(y)); the ordinariness of y is given by Ord(x), scoinc(x,y) and A39(ordinary restriction); by assumption follows ¬2D(y)(+); with scoinc(x,y) ∧ 2D(x), A28(range of coincidence) and D31(equal dimension) we get 2DB(y)(*); by using (+), (*) and D41(one-dimensional connected surface) follows ¬1DC(y), that means with D33(one-dimensional connected) there is a division in y_1 and y_2 with (eqdim(y_1,y_2) ∧ sum(y_1,y_2)=y ∧ ¬sov(y_1,y_2))(++) and furthermore **there are no one-dimensional hyper parts y_1‘ of y_1 and y_2‘ of y_2 with scoinc(y_1‘,y_2‘)**; with (++) and A41(equal cardinality) we get the existence of a division in x_1 and x_2 with (eqdim(x_1,x_2) ∧ sum(x_1,x_2)=x ∧ ¬sov(x_1,x_2))(**) ∧ scoinc(x_1,y_1) ∧ scoinc(x_2,y_2); x is one-dimensional connected, because of 2D(x)(see definition D41); by using (**) and D33‘(one-dimensional connected) we conclude the existence of one-dimensional hyper parts x_1‘ of x_1 and x_2‘ of x_2 with scoinc(x_1‘,x_2‘); with A33(existence of coincident hyper parts), scoinc(x_1,y_1) and hypp(x_1‘,x_1) we get the **existence of a one-dimensional hyper part y_1‘ of y_1** with scoinc(y_1‘,x_1‘) **and** analogous the **existence of a one-dimensional hyper part y_2‘ of y_2** with scoinc(y_2‘,x_2‘); finally with the transitivity of spatial coincidence and scoinc(y_1‘,x_1‘), scoinc(y_2‘,x_2‘) and scoinc(x_1‘,x_2‘) we get **scoinc(y_1‘,y_2‘)** and this is a contradiction

T56‘. *∀xy (1-crosssurface(x) ∧ scoinc(x,y) → 1-crosssurface(y))*

”ordinary surfaces may coincide with ordinary surfaces”

T57. $\forall x x_1 x_2 (x = sum(x_1, x_2) \wedge (\bigwedge_{i=1}^{2} 2D(x_i) \wedge Ord(x_i)) \wedge \neg sov(x_1, x_2) \wedge scoinc(x_1, x_2)$ $\wedge \neg\exists\, x‘\ (x{=}sum(x‘) \wedge 2D(x‘) \wedge Ord(x‘)) \rightarrow \neg\exists\ y\ (y \neq x \wedge scoinc(x,y))$

”no other spatial entity may coincide with a 2-crosssurface ”

Proof: reduction to the absurd; assume existence y with **y≠x** ∧ **scoinc(x,y)**; by using that x_1 and x_2 are non-overlapping spatial parts of x that are coincident and D16(extraordinary) we get **ExOrd(x)**(+); furthermore with A12(codomain sum) follows **2DB(x)**(*); with the help of A28(range scoinc),(*) and A39(ordinary restriction),(+) we derive **2DB(y)** ∧ **ExOrd(y)**; the bold marked conclusions contradict T31(no two extraordinary coincident surfaces)

T57‘. $\forall x$ *(2-crosssurface(x)* $\rightarrow \neg\exists$ *y* $(y\neq x \wedge$ *scoinc(x,y))*

"no other spatial entity may coincide with a 2-crosssurface"

T58. $\forall x x_1 x_2 y (x = sum(x_1, x_2) \wedge (\bigwedge_{i=1}^{2} 2D(x_i) \wedge Ord(x_i)) \wedge \neg sov(x_1, x_2) \wedge scoinc(x_1, x_2)$ $\wedge \neg\exists$ x‘ $(x=sum(x‘) \wedge 2D(x‘) \wedge Ord(x‘)) \wedge$ *scoinc(x,y)* $\rightarrow$ *x=y)*

"a 2-crosssurface **only** coincides with itself"

Proof: obvious, on the one hand we have reflexivity of spatial coincidence(A29) and on the other hand with T57(no other spatial entity may coincide with a 2-crosssurface) follows x=y

T58‘. $\forall xy$ *(2-crosssurface(x)* $\wedge$ *scoinc(x,y)* $\rightarrow$ *x=y)*

"a 2-crosssurface **only** coincides with itself"

T59‘. $\forall xy$ *(scoinc(x,y)* $\wedge$ *n-crosssurface(x)* $\wedge$ *m-crosssurface(y)* $\wedge$ $n,m\in\boldsymbol{N} \rightarrow$ *(n=m=1* $\vee$ *(n=m=2* $\wedge$ *x=y))))*

"main theorem"

Proof: final summary of T53, T54‘, T55‘, T56‘ and T58‘

4.6 Elementary Equivalence

In this section we want to introduce the notion of mereotopological elementary equivalence between spatial entities. Two spatial entities are elementary equivalent if and only if the same sentences (with respect to a certain signature) are true about them.

For every spatial entity E we may define the correspondent universe of discourse $\mathcal{U}$(E), structure $\mathcal{A}$(E) and theory $\mathcal{T}(\mathcal{A}$(E)).

- $\mathcal{U}$(E)= SPart(E) $\bigcup$ Hypp(E) (correspondent universe)
- SPart(E)= {x| spart(x,E)} and Hypp(E)= {x| hypp(x,E)}
- $\mathcal{A}$(E)=($\mathcal{U}$(E),spart(.,.),scoinc(.,.),sb(.,.)) (correspondent structure)
- $\mathcal{T}(\mathcal{A}$(E))={σ |$\mathcal{A}$ (E)$\models \sigma$} (correspondent theory)

Now we may define that two spatial entities E_1 and E_2 are said to be mereotopological elementary equivalent "$E_1 \equiv E_2$" if and only if they have the same correspondent theories "$\mathcal{T}(\mathcal{A}(E_1))=\mathcal{T}(\mathcal{A}(E_2))$".

Elementary equivalence is an equivalence relation like isomorphism between two structures but note that elementary equivalence is weaker than isomorphism in the sense that isomorphism implies elementary equivalence between two structures and not vice-versa. The rational numbers $\mathbf{Q}$ and the real numbers $\mathbf{R}$ with the usual less than "$<$" are elementary equivalent[Gloe 2006/07] but obviously not isomorphic (different cardinality).

The Brentanoraum $\mathbf{B}^3$ is divided into classes of equivalence by this relation. In the following subsections we want to give a first impression of the abundance of these classes.

4.6.1 Point Region

The classification of point region is less comprehensive than their higher-dimensional analogs because point regions have no boundaries, they are "built of" atoms (points) and furthermore there is no classification by connectedness.

The class of ordinary point regions is divided into points and point regions with a certain cardinality in the sense that they are the mereological sum of n ($n \geq 2$) pairwise disjoint non-coincident points. In case of extraordinary point regions we have to distinguish between the n-crosspoints ($n \geq 2$) and hybrids, whereas hybrids are the mereological sum of an ordinary point region and extraordinary cross-entities. Because of definition D15 (ordinariness) the mereological sum of an ordinary and extraordinary entity is again extraordinary.

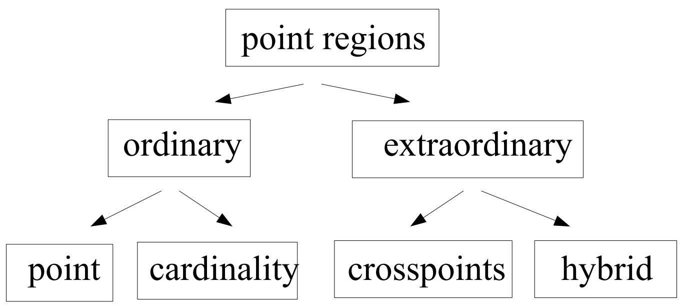


Figure 4.23: Classification of Point Regions

4.6.2 Line Region

The classification of line regions is very interesting because there are some analogies to graph theory[9]. Graphs may be interpreted as ordinary connected line regions with and without boundaries. There are a few results about decidability and undecidability in graph theory, e.g. [Her 1972] and [Her 1973]. These results may be used to show decidability or undecidability of the correspondent theories of certain line regions.

The classification of line regions is very complex. Line regions may be distinguished in ordinary and extraordinary line regions. Extraordinary line regions are crosslines and again hybrids, whereas hybrids are the mereological sum of an ordinary line region and extraordinary crossline. We want to check up the ordinary entities in more detail. Therefore we want to introduce the following two definitions:

D69. $segm(x) \Leftrightarrow 1D(x) \wedge Ord(x) \wedge \exists x_1 x_2\ (sb(x_1,x) \wedge sb(x_2,x) \wedge x_1 \neq x_2) \wedge \neg\exists x_1‘x_2‘x_3‘\ (\bigwedge_{i=1}^{3} sb(x_i‘,x) \wedge \bigwedge_{1\leq i<j\leq 3} x_i‘\neq x_j‘) \wedge \neg\exists y_1 y_2 y_3\ (\bigwedge_{i=1}^{3} hypp(y_i,x) \wedge (\bigwedge_{1\leq i<j\leq 3} y_i \neq y_j \wedge scoinc(y_i,y_j)))$

"x is a segment"

D70. $circ(x) \Leftrightarrow 1D(x) \wedge Ord(x) \wedge \neg\exists x_1\ sb(x_1,x) \wedge \neg\exists x_1‘x_2‘x_3‘\ (\bigwedge_{i=1}^{3} hypp(x_i‘,x) \wedge (\bigwedge_{1\leq i<j\leq 3} x_i‘\neq x_j‘ \wedge scoinc(x_i‘,x_j‘)))$

"x is a circle"

A segment is an ordinary connected line with exactly two boundaries and no n-crosspoints (n≥3) and a circle is also an ordinary connected line without boundaries and n-crosspoints (n≥3). Some examples are given in the following figures.

[9]Graph theory is a branch of mathematics and computer science. The solution of the "Königsberg Bridge problem"(solved by Leonard Euler) in 1736 is regarded as the beginning of classical graph theory.

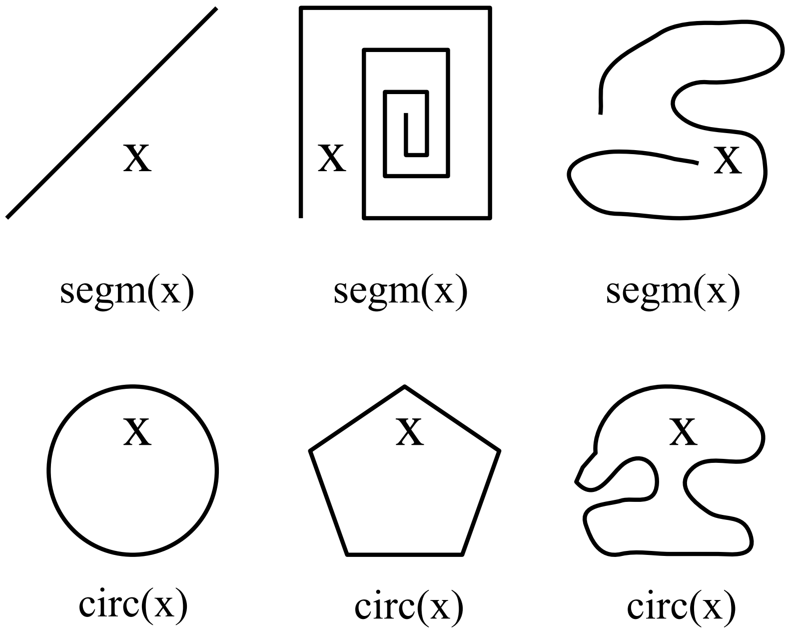


Figure 4.24: Segments and Circles

Ordinary lines may be connected or not. If they are not connected we may distinguish them by their cardinality of connected components in the sense that x is the mereological sum of $x_1,\ldots,x_n$ in which $x_1,\ldots,x_n$ are non-overlapping ordinary connected line regions[10]. Connected ordinary line regions may have boundaries (e.g. a segment) or not (e.g. a circle). In case of yes they may be classified by their cardinality of boundaries, e.g. x has exactly seven boundaries. Furthermore we may distinguish them by their cardinality of crosspoints, e.g. x has exactly two 3-crosspoints and five 6-crosspoints (compare definition D60). Note that there is a correlation between the cardinality of crosspoints and the cardinality of boundaries (see remarks in this subsection).

The information that two connected ordinary line regions have the same cardinality of crosspoints and boundaries is not sufficient for their elementary equivalence because one may ask for their cardinality of their inscribed circles in the sense that x is the mereological sum of two different circles[11] or x has two inscribed circles.

[10]In section 4.3.9 we defined connected components for three-dimensional entities. These definitions may be generalized for line regions.

[11]Furthermore it is possible to classify them by their minimal cardinality of non-overlapping circles and segments in the sense that x is the mereological sum of n circles and m segments.

Consider the following example. The ordinary connected line regions x and y are boundaryless and have exactly two 4-crosspoints but they are not mereotopological elementary equivalent.

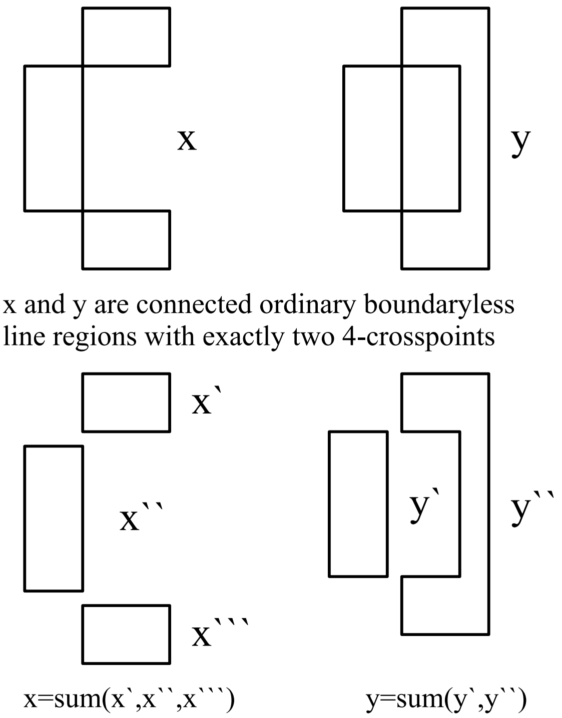


Figure 4.25: Mereotopological Elementary Equivalence (Counter-Example)

To prove the elementary inequivalence of x and y we define the following formula:

$$\varphi(x) := \exists x`x``(circ(x`) \wedge circ(x``) \wedge x` \neq x`` \wedge x = sum(x`, x``))$$

Obviously, we have $\varphi \in \mathcal{T}(\mathcal{A}(\mathrm{x}))$ and $\varphi \notin \mathcal{T}(\mathcal{A}(\mathrm{y}))$, hence the line regions x and y are not mereotopological elementary equivalent.

The following schema presents a first impression of the variety of ordinary line regions. Note that there are many possibilities to improve this schema, e.g. classification by minimal cardinality of non-overlapping segments and circles.

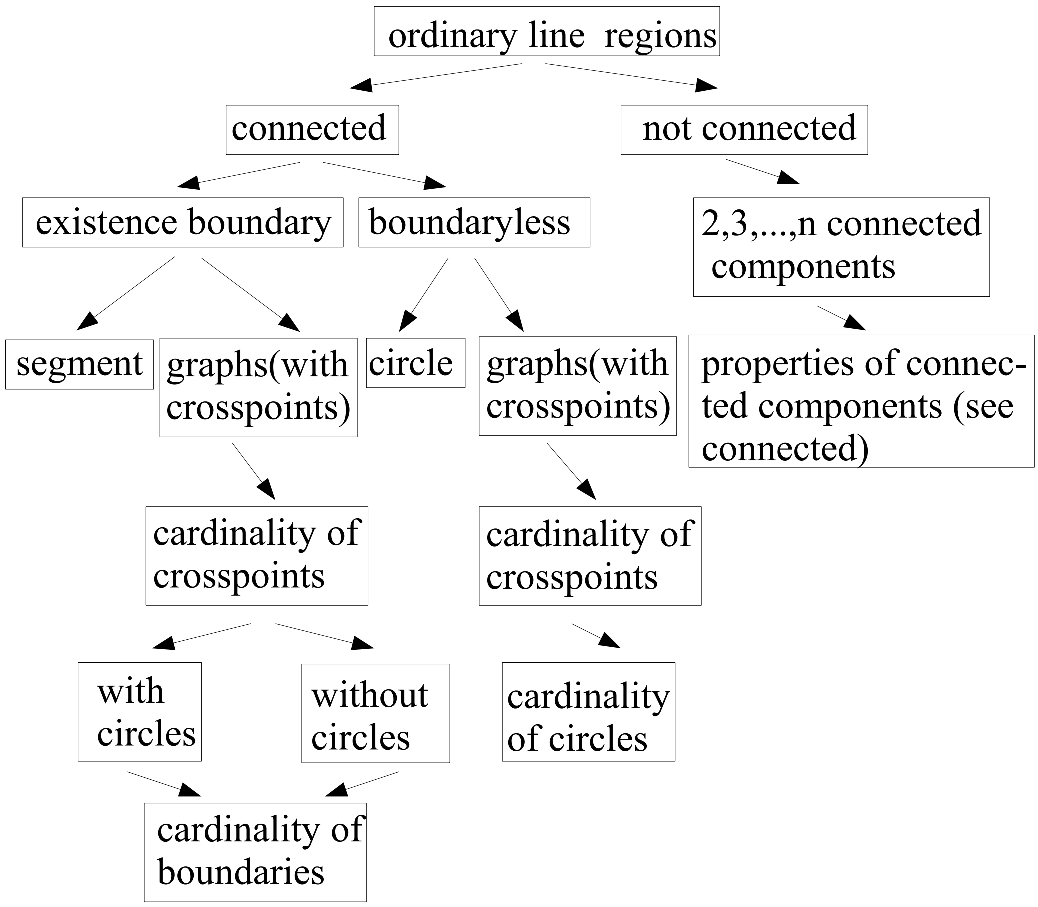


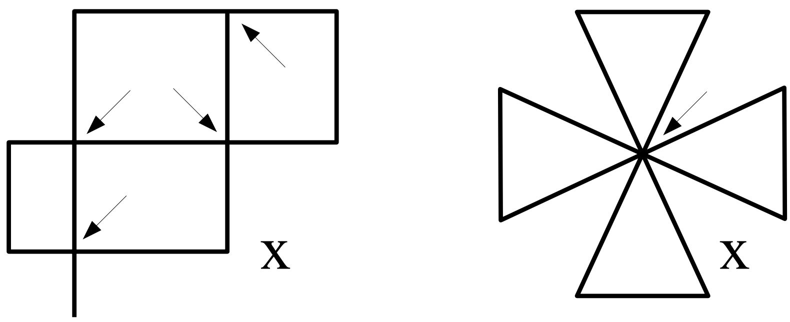
Figure 4.26: Classification of Ordinary Line Regions

Remarks about the Correlation between Crosspoints and Boundaries

In figure 4.24 we have shown that the information about crosspoints and boundaries of two ordinary connected line regions is not sufficient for their elementary equivalence. The following equations show that crosspoints and boundaries of a certain ordinary connected line region do not exist in arbitrary constellation.

It is possible to quantify the maximal (Eq1) and furthermore all possible cardinalities of spatial boundaries (Eq2) of a certain line region if the cardinality of crosspoints is given. Thus we may quantify the number of possibilities of different numbers of spatial boundaries (Eq3). These results may be useful to find conditions for elementary equivalence of line regions.

In the following x is a connected ordinary line region (=graph) and x_n the cardinality of n-crosspoints (n≥3) of x. If there is a n-crosspoint x‘ and m-crosspoint x“ with the property x‘ is spatial part of x“, then we will count only x“(see figure below). All other not given cardinalities are zero.



$x_3=1$, $x_4=3$ $x_8=1$

Figure 4.27: Cardinality of Crosspoints

With the help of this agreement we may give the following equations:

Eq1. $sb_{max}(x)=2+\sum_{k=3}^{\infty} x_k(k-2)$

"maximal number of spatial boundaries"

Eq2. $sb_i(x)=2(1-i)+\sum_{k=3}^{\infty} x_k(k-2) \qquad 0\leq i\leq\left\lfloor \frac{sb_{max}(x)}{2}\right\rfloor$

"possible numbers of spatial boundaries"

Eq3. $card\ \{sb_i(x)=\left\lfloor \frac{sb_{max}(x)}{2}\right\rfloor+1$

"number of possibilities"

The first equation may be easily verified by combinatorial consideration. The second and third are obvious because the second reduces the maximal number incremental about two (up to zero or one) and the third equation only counts this possibilities.

We want to give an example to underline the notion of the equations. Consider an arbitrary graph x with one 3-crosspoint and three 4-crosspoints. With equation 1 results $sb_{max}(x)=2+\sum_{k=3}^{\infty}x_k(k-2)=2+1(3-2)+3(4-2)=9$, that means the maximal number of spatial boundaries of such a graph is nine. By using equation 2 we conclude that $0\leq i\leq 4$, because the floor function[12] of $\frac{9}{2}=4$, therefore we have $sb_0=sb_{max}=9$, $sb_1=7$, $sb_2=5$, $sb_3=3$ and $sb_4=1$, that means there are five possibilities of cardinality of spatial boundaries.

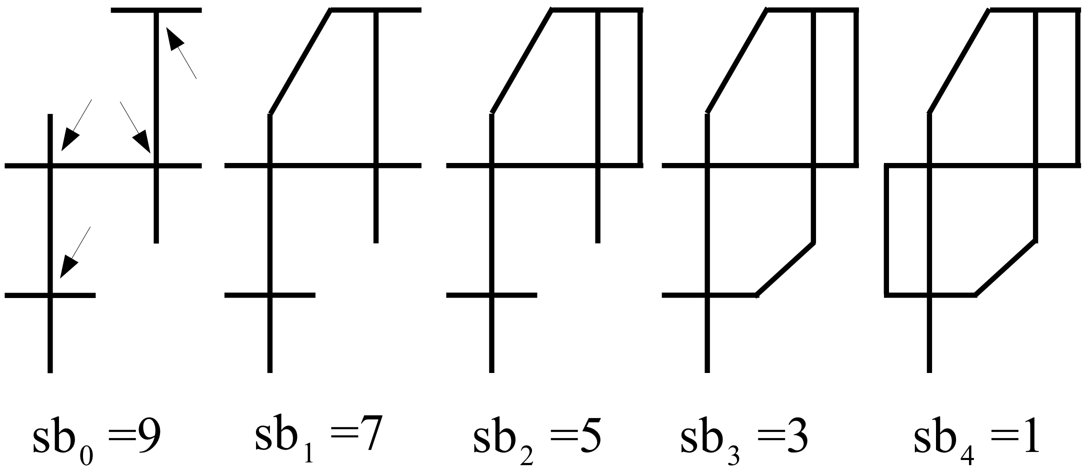


Figure 4.28: Equation-Example

4.6.3 Surface Region

The classification of two-dimensional entities is again more complex than their lower-dimensional analogs because beside ordinariness and the existence of boundaries we have to distinguish between two kinds of connectedness, namely zero- and one-dimensional connectedness (compare figure 4.12).

Non-connected surface regions may be classified by their connected components and the corresponding properties of their components. Because of the existence of two kinds of connectedness we have several options to define connected components for surface regions. One may count the minimal cardinality of non-overlapping one-dimensional or zero-dimensional connected ordinary surface regions. Furthermore it is possible to count the minimal combination of them in the sense that x is the mereological sum of exactly 3

[12]The floor function maps real numbers to the next lower integer.

one-dimensional and 2 zero-dimensional connected ordinary surface regions.[13]

Extraordinary surfaces are divided in 2-crosssurfaces and hybrids, whereas hybrids are the mereological sum of 2-crosssurfaces and an ordinary surface region. Note that 2-crosssurfaces are the only kind of extraordinary two-dimensional cross-entities (compare theorem T53 and T54). Hybrids may be classified by their cardinality of inscribed 2-crosssurfaces and furthermore by the properties of them and the properties of their ordinary component.

The results of Andrzej Grzegorczyk [Grze 1951] give reason to believe that all corresponding theories of two-dimensional spatial entities are undecidable (future work). He had shown that the theory of several topological spaces is undecidable.

The following schema gives a first impression of the diversity of surface regions. For reasons of clear arrangement we will exclude the distinguishing feature "existence of crosslines" (compare definition D61). In section 4.6.4 we will explain a further possibility of classification (genus).

[13] It is easy to show that the following inequality holds for every surface region x. Cardinality of one-dimensional connected components(x)$\geq$Cardinality of zero-dimensional connected components(x).

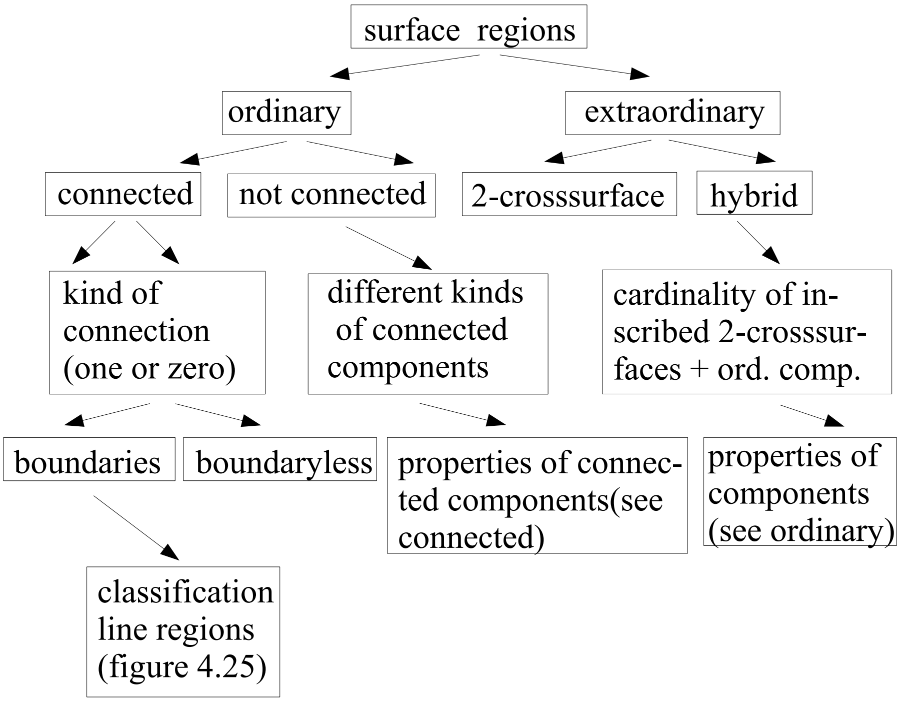


Figure 4.29: Classification of Surface Regions

4.6.4 Space Region

The three-dimensional spatial entities are divided in connected and not connected space regions. In subsection 4.3.6 we defined three kinds of connectedness, namely two-, one- and zero-dimensional connected. Therefore we got three different possibilities to define connected components (see definitions D67 and D68).

Space regions are ordinary per definition. We may classify them by their connected components. Furthermore we may ask about properties of their spatial boundaries or the properties of the spatial boundaries of their spatial boundaries and so on. That means the classification of space regions is the most complex one.

Is it possible to distinguish the occupied space region of a scoop of ice cream and a doughnut? The maximal two-dimensional boundaries of them are a 2-sphere and a torus. These surfaces may be distinguished in mathematical topology by their genus[14]. The genus of a surface may be defined in simple terms as the number of holes or handles in it.

[14]The genus of a surface is closely related with the Euler characteristic. Both are topological invariants.

Consider therefore the following examples:

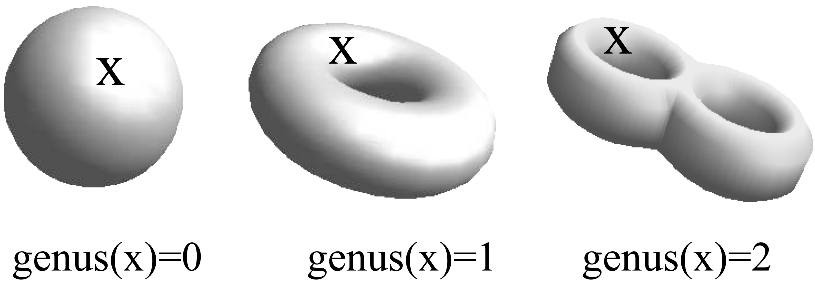


Figure 4.30: Genus-Examples (Sphere, Torus, Pretzel)

The following question arises: Is it possible to show that such spatial entities are not mereotopological elementary equivalent? That means we want to distinguish them without adding a new (morphological) basic relation like "genus of x" or "x is a hole". A possible distinguishing feature for a 2-sphere and a torus is the following formula φ(x):

$$\varphi(x) := \forall y((circ(y) \wedge hypp(y,x)) \rightarrow \exists x`(spart(x`,x) \wedge maxb(y,x`)))$$

The formula φ(x) expresses that for every circle y on a spatial part x‘ of x exists with a maximal boundary which is equal to y. It is obvious that $\varphi \in \mathcal{T}(\mathcal{A}(\text{2-sphere}))$ and we will show that $\varphi \notin \mathcal{T}(\mathcal{A}(\text{torus}))$. Consider therefore the following figures:

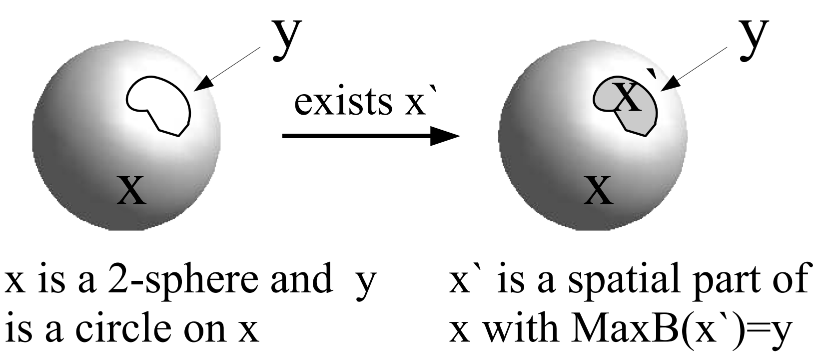


Figure 4.31: Circle-Example (2-Sphere)

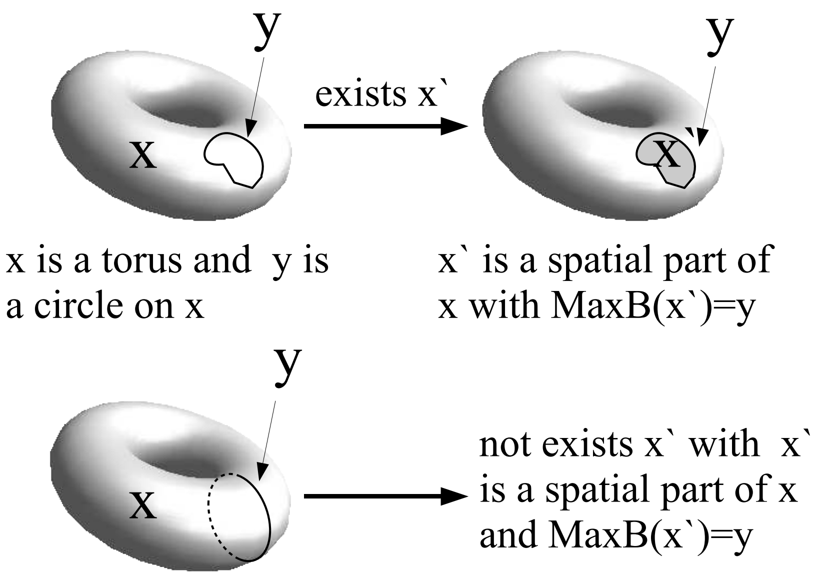


Figure 4.32: Circle-Example (Torus)

The first example in figure 4.31 shows that there are circles y on torus x and spatial parts x‘ of torus x with the property that y is the maximal boundary of x‘. The second example demonstrates that φ(x) is not true in general. Note that it is possible to find a spatial part x‘ with the property that y is a spatial boundary of x‘ (but not maximal).

With the help of this formula we may distinguish the occupied space region of a scoop of ice cream and a doughnut because one has to apply φ(x) to the maximal boundary of the occupied space regions in the following way:

$$\varphi`(x) := \forall y((circ(y) \wedge hypp(y, MaxB(x))) \rightarrow \exists x`(spart(x`, MaxB(x)) \wedge maxb(y, x`)))$$

The formula φ‘(x) is useful to distinguish a three-dimensional ball and a filled torus but it is not sufficient to characterize a three-dimensional ball because φ‘(x) holds for cubes, too. Maybe it is possible to define morphological notions like x is a disc, a rectangle, a cube, a torus or a pretzel in a mereotopological theory without adding a new basic relation. Tarski figures out that there is a defining property for discs or balls, namely: Every mereological relative complement of two different overlapping discs (balls) is connected (see figure below).

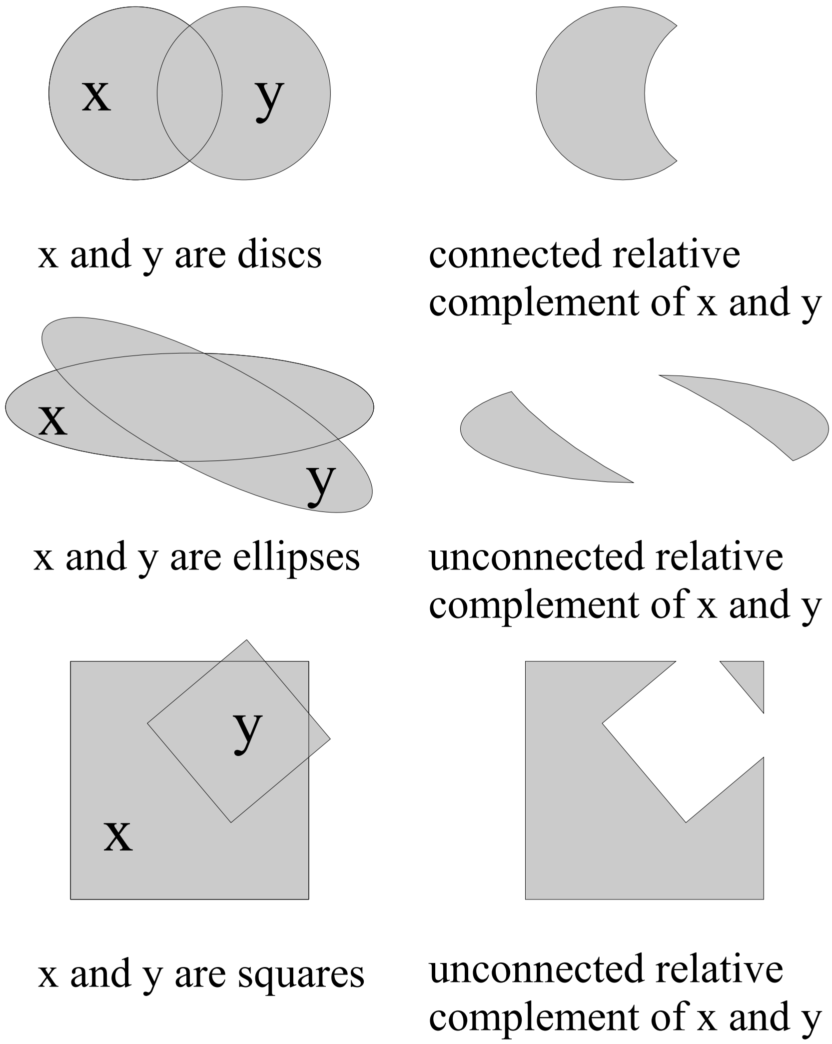


Figure 4.33: Defining Property of Discs (Balls)

Further distinguishing features of space regions may be holes, graves and tunnels. A good overview about this kind of classification is given in [Cas, Var 1994].

The following schema gives a basic classification of space regions. Note that we mentioned many possibilities to improve this classification, e.g. defining property of balls or the genus of space regions.

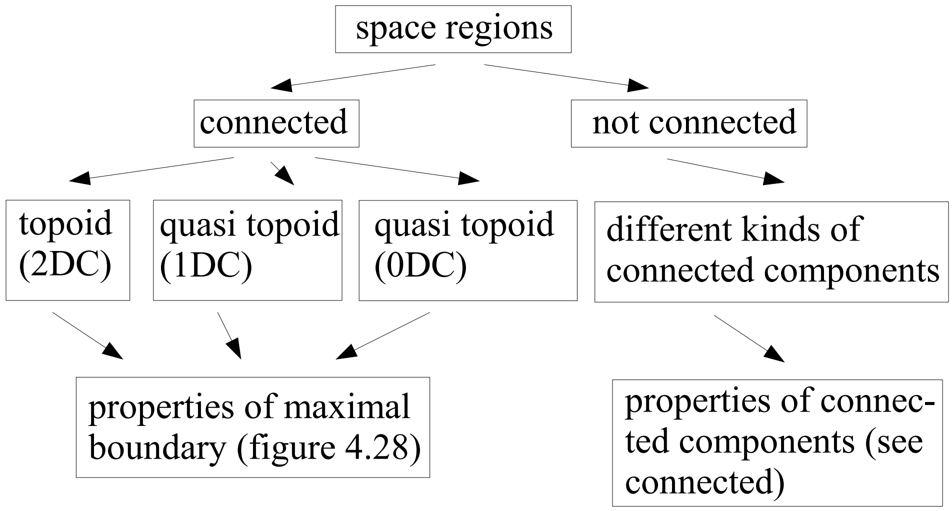


Figure 4.34: Classification of Space Regions

Chapter 5

Material Entities

5.1 Preliminary

Consider the following famous question of Leonardo da Vinci:

> *What is it... that divides the atmosphere from the water? It is necessary that there should be a common boundary which is neither air nor water but is without substance, because a body interposed between two bodies prevents their contact, and this does not happen in water with air.*" (cited in [Cas, Var 1994]).

In this chapter we will present an axiomatic theory of material structures which may explain and answer questions like this. At first we want to remember our philosophical point of view of material structures because if you want to explain contact between material structures or even qualities like color or state of aggregation of them, you have to explain what a material structure is.

According to our explanations in chapter 2 we want to distinct between the urobject ("thing-in-itself") that we call a *material structure per se* and its belonging dispositions like extension, form and substance. Phenomenal objects can be understood as a set of unfold dispositions (=attributes) of the urobject. In our considerations we will use the term *material structure* for a certain phenomenal object. Keep in mind that a phenomenal object cannot be separated of the perceiving subject. The following illustration summarizes these interrelations.

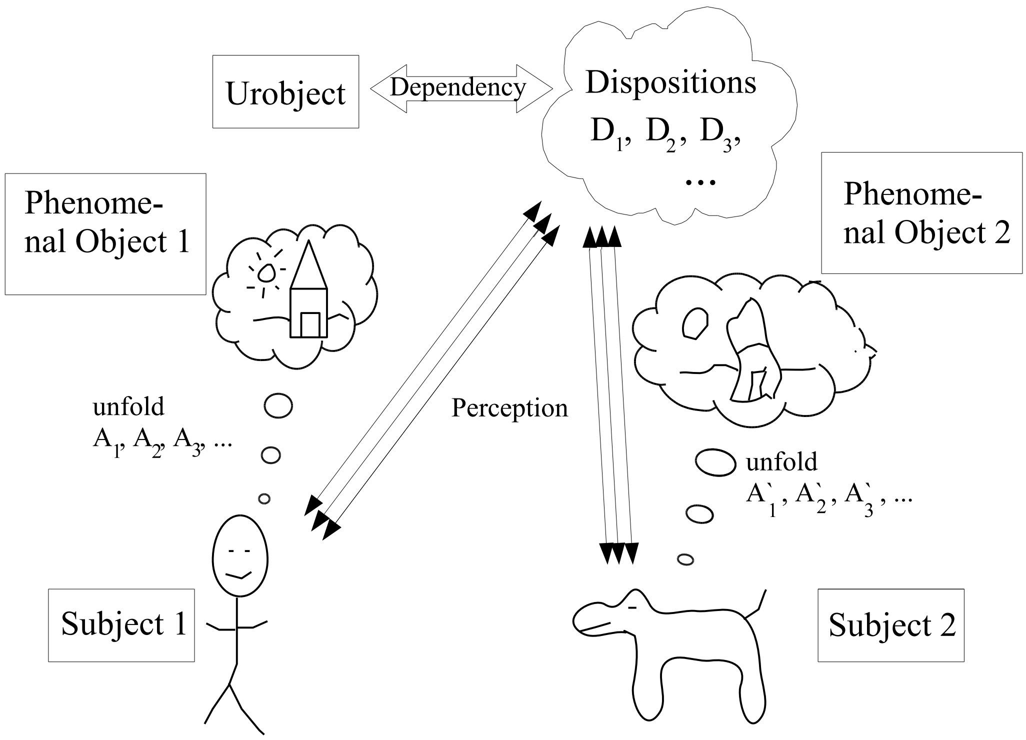


Figure 5.1: Phenomenal Objects (House-Illustration)

5.2 Material Structures

According to our Top-Level-Ontology GFO [Her, Hel, Bur, Hoehn, Loe, Mich 2006] we want to assume that a material structure is an individual that fulfils the following conditions: it is a presential, it is a bearer of qualities , it occupies space and it consists of a presential amount of substrate. Consider therefore the following sentences:

1. The analysis of the photo finish revealed that **Carl Lewis** passed the finish line at first. (wholly present at a certain time-point)

2. Normally the **sky** is blue but during the sundown it is colored in red. (quality color and different values)

3. The weight of an **astronaut** on the moon is only the sixth part of the weight on the earth. (quality weight and different values)

4. The expensive **Ferrari** of my neighbor is parked in his **garage**. (occupied space and location)

5. The electron microscope shows that **water** consists of molecules that are arranged of two hydrogen atoms and one oxygen atom. (occupied space and granularity)

6. Last christmas the **pond** was frozen. (presential amount of solid substrate)

5.2.1 Presentialist Character

A *presential* x is an individual which is wholly present at a certain time-point t. In GFO we introduced the relation *at(x,t)* which describes this situation. It can be understood as a projection operator. The contrary of a presential is a process which has a temporal extension like a tennis tournament. It is obvious that a tournament cannot wholly be present at a single time-point.

The presentials Carl Lewis at the time-point t_1 and Carl Lewis at the time-point t_2 ($t_1 \neq t_2$) are different in the sense that both have different qualities like location or hair length but they are equivalent in the sense that both are instances of the same persistant[1]. That means we want to distinguish between the universal "Carl Lewis" that persists through the time and the instances of it which are presentials.

5.2.2 Bearer of Qualities

A material structure is a bearer of qualities like an individual color or a certain distance to another object but it cannot be a quality of other entities. To clarify the situation between the material structure and their qualities we have to introduce the distinction between abstract *properties* and their *property values* and individual *qualities* and their *quality values* at first.

Properties and property values are universals and qualities respective quality values are instances of them being individuals. Consider the following sentences: "The color of this sky is red." We can differentiate between four entities, namely: 1. "the color" (property), 2. "the color of this sky" (instance of property = quality), 3. "red color" (property value) and 4. "the red color of this sky" (instance of property value = quality value). The instances of the universals cannot be separated of the material structure. To capture the situation between the material structure x and its belonging quality y, we will define a

[1]More precise Carl Lewis is a perpetuant that exhibits different presentials at different time-point. A detailed consideration is given in [Her, Hel, Bur, Hoehn, Loe, Mich 2006] subsection 6.2

relation of inherence denoted by *inh(x,y)* with the meaning "the quality x inheres in the material structure y".

Qualities may be distinguished by their type of perception, namely visually (color), auditory (acoustics), tasty (rugosity), olfactory (strong-smelling) and gustatory (sweetness) qualities. Note that perceptible qualities are not independent of the granularity level. This will be discussed in subsection 5.7.3.

Furthermore we introduce the notion of *measurement systems*. Measurement systems are a set of possible property values of a certain property. For instance, weight may be measured with the real numbers and a certain unit or distance may be measured with values like "closed to" or "far away".

5.2.3 Occupied Space

Material structures have the ability to occupy space (spatial location). This ability is caused by the intrinsic quality to have a spatial extension, which is called the *extension space*. The extension space is a quality like the individual color or weight of a material structure and that is the reason why a particular extension space cannot be shared between two different material entities.

We will use the relation *occ(x,y)* with the meaning "the material structure x occupies the space region y". Note that this relation is not used in a maximal sense as well as the relation "spatial boundary" in the Brentanoraum. That means there can be another space region y‘ which is occupied by the same material structure x with the property that y is a spatial part of y‘. The maximal variant will be defined as *maxocc(x,y)* analog to the maximal boundary in the Brentanoraum. Consider therefor the following figure.

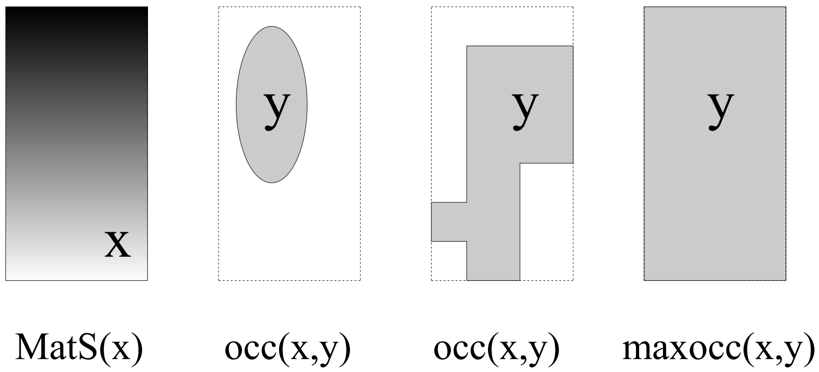


Figure 5.2: Occupied Spaces of a Material Structure

The occupied space entities of a certain material entity depend on granularity or distance and we will assume that the maximal occupied space is uniquely determined for every granularity level. Note that this axiom is not unproblematic because in subsection 5.7.2 we will see that there are several possible interpretations of the maximal occupied space of a certain material entity even if the granularity level is fixed.

5.2.4 Amount of Substrate

It is obvious that every material structure consists of an amount of substrate. Analog to the subsection 5.2.1 we have to take into consideration that the amount of substrate has a persistant and presential character. Imagine that you put an ice-cube in a glass. After five minutes the ice has melted. On the one hand it is the same amount of substrate because we did not exchange the ice cube for liquid water. On the other hand they are different because they have different states of aggregation.

To capture the situation between the persistant and its presentials we will introduce the basic relations "Substr(x)" and "PSubstr(x)" with the meaning "x is a persistant amount of substrate" and "x is a presential amount of substrate". The presential amount of substrate may be understood as an instance of the persistant amount of substrate at a certain time-point.

In our axiomatization we will distinguish between solid, gaseous and liquid substrates.

Note that these are not all states of aggregation[2] but they are the most important in daily life.

5.3 Material Boundaries

5.3.1 Perception and Essence

The perception of material structures is in the first instance a perception of their belonging material boundaries which demarcate them from their surroundings. That means if you look to a material structure like a dice or a house, you will see their belonging surface areas at first and secondly a three-dimensional object (by a cognitive act).

What is the essence of material boundaries? We assume that material boundaries are dependent entities that means they cannot exist in isolation. Every material boundary belongs to a certain higher-dimensional material entity. Furthermore we assume that they are cognitive items which do not belong to the physical level of reality. That means material boundaries are not three-dimensional material parts of a material structure but rather lower-dimensional entities without an amount of substance.

5.3.2 Non-Coincidence

According to spatial boundaries we want to distinguish between *material surfaces, material lines* and *material points*[3]. Every material surfaces is a material boundary of a certain material structure. The same holds for material lines and points that means there are material boundaries of a certain material surface or line.

The main distinguishing feature between spatial and material boundaries is that spatial boundaries can coincide and material boundaries cannot. That means spatial and material boundaries belong to different ontological categories. The reason for this is the following: The consideration of two material cubes placed side by side as a material aggregate (compare section 5.6) do not imply the material connectedness of the aggregate. But if we assume the possibility of coincidence of material boundaries, we have to follow

[2]The state of aggregation depends on temperature and pressure. Under extreme conditions it is possible to generate suprafluids (has no internal friction) or plasma (for example stars).

[3]Keep in mind that material boundaries are not "material" in the sense that they consist of an amount of substrate. They are cognitive items.

the material connectedness of the aggregate because its boundaries are not only placed side by side but rather co-located.

Contact between two material structures will be defined as follows: Two non-overlapping material structures are in contact (touching each other) if and only if there are at least two material boundaries of them which occupy coincident spatial boundaries.

5.3.3 Induced Qualities

Material boundaries may have qualities like color or shape. At the beginning of this section we declared that the perception of a certain material structure is predominantly a perception of their belonging two-dimensional material boundaries. That is why we want to introduce the notion of "induced qualities" of a material structure by their material boundaries. Consider a red bowl. In daily life we say that the bowl is red because the surface area of the bowl is red. Furthermore we say that the bowl is shaped like a ball because the perceived material boundary is shaped like a sphere. Note that there are qualities of material structures that are independent of the qualities of their material boundaries, for instance weight, specific gravity or a presential amount of substrate.

With the help of induced qualities we may distinguish between a bowl with a red surface and a bowl with a red presential amount of substrate. The second case implies the first case but not vice versa.

5.3.4 Interrelations between Spatial and Material Boundaries

The connection between material boundaries and the Brentanoraum $\mathbf{B}^3$ is the following: Every material boundary x‘ of a material structure x occupies a spatial boundary or hyper part of the maximal occupied space region of x. The condition that x‘ may occupy a hyper part of the maximal occupied space region of x cannot be removed. Consider therefor the "Mathias-example" below. The head and the hand of Mathias have their own material boundaries. These boundaries do not disappear if he touches his head with his hand. The maximal occupied space region of Mathias is spatial connected. This means that there are material boundaries of his head or hand that do not occupy a spatial boundary ("outer boundary") but rather a hyper part ("inner boundary") of the maximal occupied space of Mathias.

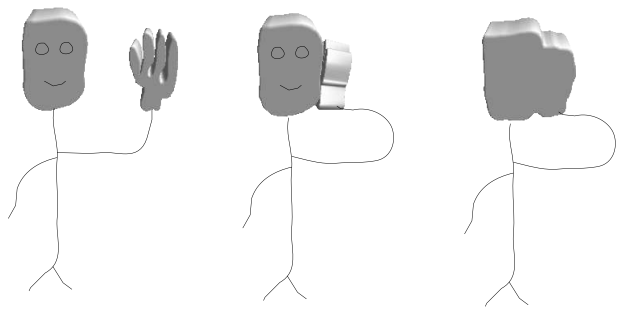


Figure 5.3: Material Boundaries (Mathias-Example)

There are at least two possibilities for the appearance of this phenomenon: 1. A material structure is in contact with itself. 2. The consideration of two material structures which are in contact as a single material structure (a so-called material aggregate[4]. The maximal material boundary of such kinds of material structures occupy extraordinary spatial boundaries. That is why we analyzed extraordinary entities very detailed in the Brentanoraum.

What we do not assume is that for every spatial boundary y‘ of the maximal occupied space y of a certain material structure x exists a material boundary x‘ of x which occupies y‘. The reason for not assuming this is that spatial entities may occur in higher variety than their material analogs. Spatial two- and one-dimensional boundaries always have spatial proper parts. Material boundaries cannot be reduced arbitrary because we fix the granularity level for every single consideration. The occupied spatial boundaries of a material boundary depend on granularity and context (see subsection 5.7.1).

5.4 Material Parts

A material part of a material structure or boundary is again a material structure respective boundary (equal dimension). Just like in case of material boundaries we will not assume that for every spatial part y‘ of the occupied space region y of a material structure x exists a material part x‘ of x that occupies y‘. The reason for this is again the higher variety of spatial entities in comparison to material entities. Space regions always

[4]compare section 5.6

have spatial proper parts (no atomic space regions) and material structures may consist of non-divisible atoms. That is why we introduce the relation "mpart(x,y)" as a new basic relation.

Material parts may be divided in conceptual and formal material parts. The conceptual material parts of a table are its four table legs and its table board. A formal material part of this table is for instance the aggregation of a half of one table leg and the half of the table board. In simple terms one may say that formal material parts are all possible (imaginable) parts of a certain material structure. Obviously, the set of conceptual material is a proper subset of the set of formal parts. That is why we want to use the term "material parts" in the general version.

The interrelation between material structures, material parts and granularity will be discussed in subsection 5.7.1.

5.5 Material Connectedness

The spatial connectedness of the maximal occupied space region of a certain material structure x is only a necessary (and not sufficient) condition for the material connectedness of x. Imagine therefore a brick and two different material cubes (considered as an aggregate) placed side by side:

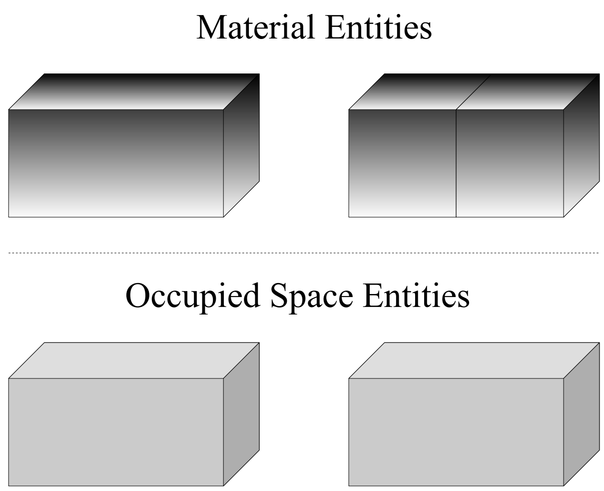


Figure 5.4: Spatial and Material Connectedness

The brick is a material connected entity and its belonging maximal occupied space is spatial connected. The cube-aggregate has a spatial connected maximal occupied space region, too but it is not material connected.

The definition of the term "material connectedness" is incomparably more difficult than its spatial analog. The reason for this is that we use the term in a very broad sense. Consider therefore the following examples:

1. A block of iron

2. A clinker construction

3. A screwed "Ikea-rack"

4. Two adjoining parts of a necklace

The similarity of all examples is the inseparability of the material structure. The examples show that there are various reasons for inseparability. The iron-block is inseparable because of the strong cohesive forces between its atoms. This cohesive attraction is a quality of its presential amount of solid substrate. The clinker construction represents all kinds of glued connections. The glue (e.g. cement) connects two materials because of the surface adhesion. This kind of connection is caused by molecular forces. The screwed "Ikea-rack" represents different kinds of screw coupling and nail connection. Those kinds of connection are caused by pressure and friction forces between the material and the screw or the nail. The necklace-example illustrates another possibility. The reasons of its inseparability are neither atomic or molecular forces nor pressure or friction forces between the materials, but rather the impenetrability of solid bodies.

The present work restricts the consideration of material connectedness to solid bodies [5]. We want to introduce the basic relation *MC(x)* with the meaning "x is material connected" in the sense of the first, second and third example. The reason of this is that we want to postulate that material connectedness implies spatial connectedness. The differentiation between all different kinds of material connectedness is future work.

[5] In case of fluids or gases we have to develop other terms (future work). Note that fluids and gases have bond forces on the atomic or molecular level, too.

5.6 Material Aggregates

A material aggregate (=material sum) is a summarization of different material structures or boundaries like the mereological sum of spatial entities. The formal definition of the material sum is analog to the spatial sum. We want to mention an important difference between the theory of spatial and material entities. In subsection 4.5.3 we exemplified that the mereological sum and the maximal boundary function are not commutative (see figure 4.22). The reason for this is that two coincident spatial boundaries may "disappear"[6] by building the mereological sum of two external connected spatial entities. If we consider the same situation on the material level[7] we observe that the building of the material sum does not "eliminate" material boundaries. That means the material sum and the maximal material boundary function are commutative in this situation in contrast to the spatial operators.

Note that the material aggregate is a theoretical construct without any restrictions. That means we may consider the material sum of the Eiffel Tower and a packet of cigarettes just like the summarization of all houses in a certain neighborhood.

5.7 Granularity

5.7.1 Granularity Function and Corresponding Sets

Every material structure has a natural granularity level or a certain "living space". A rough classification of granularity levels is the following: cosmological, macroscopical, mesoscopical and microscopical world. Examples of "life-forms" are sequentially given by saturn and earth, human beings and everyday objects, cells and proteins and on the microscopical level for instance electrons and neutrons. A more precisely classification may be given by different size ranges like "1-5 meters maximal diameter" or "5-100 million miles maximal diameter".

In some situations it is useful to consider a material structure on a lower granularity level than its natural. The atomic structure of saturn cannot be investigated on the cosmological level as well as questions of different kinds of cells of a human organism on the

[6]The spatial boundaries do not disappear in the sense that they do not exist anymore but they switch from "outer boundaries" (=spatial boundaries) to "inner boundaries" (=hyper parts).

[7]Two material squares (or cubes) placed side by side

macroscopical level.

Whatever granularity level is interesting we want to assume that the granularity level is fixed for every single consideration and therefore the granularity of the belonging material parts. That means the Atlantic Ocean or the Antarctic are material parts on the cosmological level of the earth in contrast to three atoms of the inner core of the earth which belong to the microscopical level. Different granularity levels induce different material boundaries of a material structure. Obviously the observed lunar surface from the earth is different to the observed lunar surface on the moon. That means different granularity levels imply different material boundaries.

Let G be a set of granularity levels, MS the set of all material structures and MS_G the set of all material structures interpreted at a certain granularity level. With the help of this agreements we may define the *granularity function.*

$$gf : MS \times G \rightarrow MS_G \quad gf(x, g) = x_g \quad \text{"}\textbf{granularity-function}\text{"}$$

The granularity function is not defined for every g∈G because you cannot observe a material structure on a higher granularity-level than their natural. What is an atom on the cosmological level or how to observe a human being in 100 million miles distance? One may define the set of all possible granularity levels of a certain material structure to avoid this problem. With the help of the granularity function one may define the corresponding sets of material parts MP(x_g) and material boundaries MB(x_g) in the following way:

- MP(x_g):= {x | x is a material part of x_g with granularity g }

  "corresponding material parts"

- MB(x_g):= {x | x is a material boundary of x_g with granularity g }

  "corresponding material boundaries"

5.7.2 Location

In relation to granularity levels one may observe different occupied space regions of a material structure. The finer the granularity level is, the more detailed is the observed space region. Imagine a train ride. One mile before entering the station you observe a big building beside the station. By losing distance to the station (=higher granularity) you may perceive more and more details of the building and therefore a more precisely occupied space region of the building.

The determination of the maximal occupied space of a certain material structure is intimately connected with vagueness. Consider a crowd on the Alexander Platz in Berlin. What is the maximal occupied space of this crowd? The following figures show some possibilities[8].

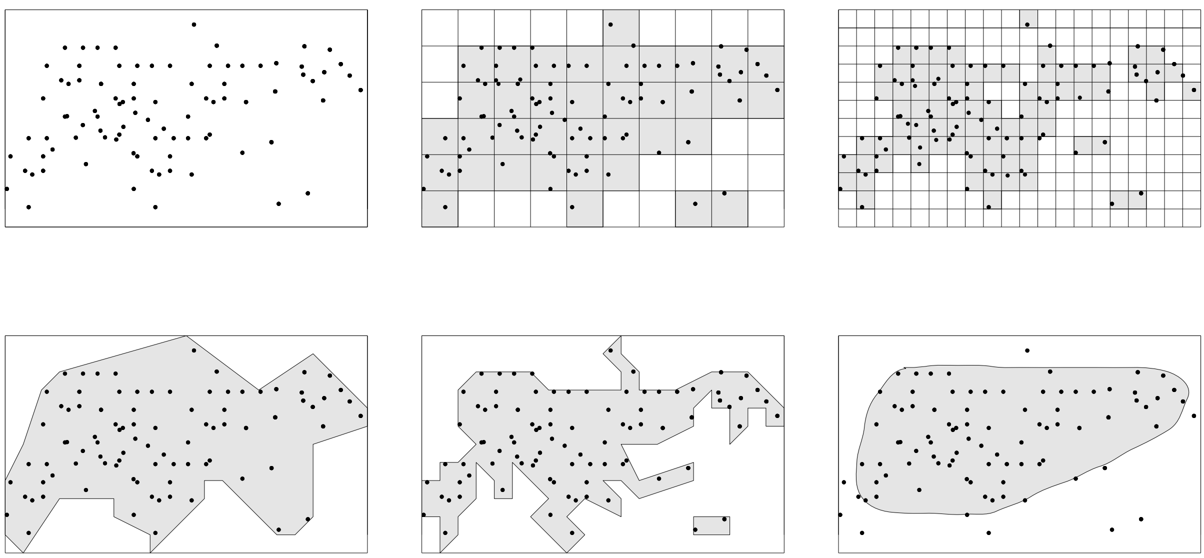


Figure 5.5: Occupied Space (Crowd-Example)

The examples above are only a small selection of all possible interpretations of location or maximal occupied space of a material structure. The figure top right may be interpreted as the finest granularity level because the occupied space of the crowd is the mereological sum of the occupied spaces of every single person. The following two figures in the upper row are working with several size ranges[9]. The first two figures in the lower row may be interpreted as several granular closure operators. The last figure is a good example

[8]For reasons of presentability we will interpret the crowd as a two-dimensional object.

[9]The condition for occupation is the following: A square is occupied by the crowd if at least one person is "in" there. This condition may be modified by the cardinality of persons.

for vagueness and maybe it represents the best approximation of the common-sense interpretation of the maximal occupied space of this crowd. Whatever kind of occupation one may choose, we want to assume that the maximal occupied space region of a certain material structure is uniquely determined for every granularity level. A detailed overview about spatial granularity is given in [Schmi 2004].

5.7.3 Perception of Qualities

In subsection 5.5.1 we introduced natural granularity levels of material structures. What about qualities of a certain material structure? Is there a correlation between the perception of a quality and the considered granularity level? Definitely yes! Consider a chessboard. At a distance of one meter we perceive a bicolored (black and white) material structure. By increasing the distance to the chessboard we perceive a continuous mixing of these colors. Finally by overrunning a certain distance we get the perception of a monochromatic (grey) material structure.

The chessboard-example shows that qualities may change under different granularity levels. Furthermore it is obvious that we cannot perceive qualities of a certain material structure if we cannot perceive the material structure itself. That means the natural granularity level of a certain material structure is a upper bound for the perception of its belonging qualities. Is there a lower bound? Consider again the quality color. Visible light has a wavelength range of 380-780 nm (nanometer). That means if we observe a certain material structure in size ranges below 380 nm, then it is just impossible to perceive a color.

The shape is another example for changing quality values by switching the granularity level. Imagine a space travel to the moon. If you leave the earth's atmosphere you will perceive an almost perfect ball. By reducing the distance you will observe more and more details of the moon-shape like mountain ranges and channels.

Analog to the corresponding sets of material parts and boundaries at a certain granularity level g, one may define the corresponding sets of perceptible qualities and their belonging values. Remember that x_g is the application of the granularity function to a certain material structure x and granularity level g.

- $Qu(x_g):= \{x \mid x$ is a perceptible quality of $x_g \}$

"corresponding qualities"

- $QuV(x_g):= \{x \mid x$ is the perceived/measured quality value of $y \wedge y \in Qu(x_g) \}$

"corresponding quality values"

5.7.4 Interrelations between Granularity Levels

Corresponding Sets

We postulated that for every single consideration the granularity is fixed. One may ask about invariants of granularity shifts. It seems to be the case, that there are no perceptible invariants if we allow all kinds of size ranges[10]. We have seen that the set of corresponding material parts, boundaries, qualities and quality values are changing under different granularity levels of a certain material structure as well as the occupied space region of it.

What happens if we reduce or increase the granularity level by a small unit? One may answer that the corresponding sets are equal. This assumption seems to be true for one granularity shift. Does it also works for many (infinitely) reducing/increasing granularity shifts. These coherences have to be investigated in more detail (future work). The works of Jerry Hobbs [Hob 1985] and Jerome Euzenat [Euz 1995] provide interesting contributions to this topic.

Maximal Occupied Space and Equality

In subsection 5.9.5 we will define "g-equality" (see definition D103) of two material structures x and y which circumstate the situation that both have the same maximal ocuppied space with respect to the granularity level g. Consider therefore the following figure:

[10]Intrinsic qualities like extension-space or the existence of boundaries are invariant but we want to focus on their "values". That means what are the material boundaries or what is the maximal occupied space?

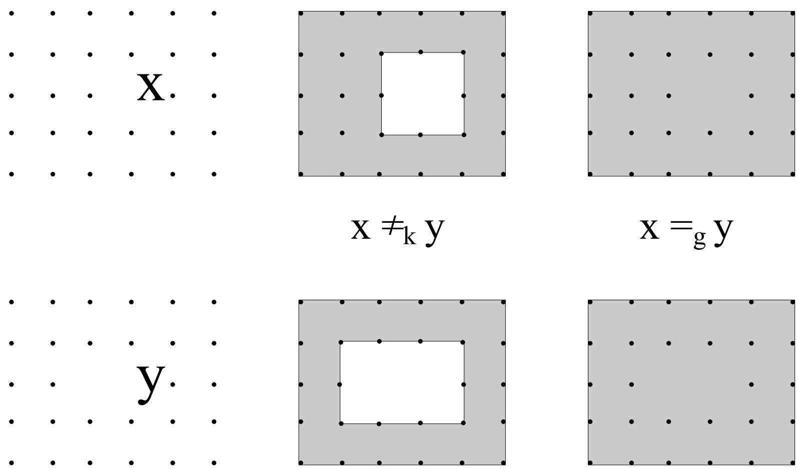


Figure 5.6: g-Equality

The figure above shows that two different material entities which may have the same maximal occupied space with respect to a certain granularity level g. Furthermore it is obvious that g-equality for one granularity level g is not sufficient for the equality of two material structures. The following question arises: Is it sufficient for the equality of two material structures x and y that x and y are g-equal for all granularity levels g? A positive answer would imply that two different material structures cannot be co-located at the same time. Note that there is no consensus about this philosophical question.

The classical example of two co-located material structures is that of a vase and the amount of clay. The justification of their difference is that they have different properties. Consider therefore the following citation:

> "...necessarily, the vase does not survive a radical change in shape or topology, while, necessarily, the amount of clay does. Therefore the two things must be different, yet co-located: as we shall see, we say that the vase is constituted by an amount of clay, but it is not an amount of clay..."(cited in [Gua, Mas, Olt, Schnei 2003] p.9)

This assumption leads to infinite objects which are co-located at the same time because their difference is justificated with different properties. Hence, the consideration of the "same" vase with an additional property leads to three different objects. This fact is dissatisfying because the assumption that there are infinite different objects, if we have at least one object, seems to be contra-intuitive.

Note that a serious talk about the problem above has to consider all levels of granularity at the same time. The present work is dealing with a fixed level of granularity for every single consideration. We have seen (figure 5.6) that two different material structures may be co-located at the same time with respect to a certain granularity. In this sense it is possible that two different structures are at the same space-time.

5.8 Basic Relations

According to our explanations in the subsections above we want to fix the granularity level for every single consideration. By using the granularity function we get the corresponding sets of material parts, boundaries, qualities and their belonging values of a certain material structure. This means that basic relations like "mpart(x,y)" or "mb(x,y)" are operating on these predetermined sets. Higher- or lower-granular parts and boundaries are excluded. In the following we will notate "x" instead of "gf(x,g)" or "x_g".

5.8.1 Presential and Material Relations

B5. *Pres(x)* "x is a presential"

B6. *MatS(x)* "x is a material structure"

B7. *mpart(x,y)* "x is a material part of y"

B8. *mb(x,y)* "x is a material boundary of y"

5.8.2 Properties and Qualities

B9. *Prop(x)* "property x"

B10. *PropV(x)* "property value x"

B11. *Qual(x)* "quality x"

B12. *QualV(x)* "quality value x"

B13. *hprop(x,y)* "material entity x has property y"

B14. *propv(x,y)* ”x is a property value of the property y”

B15. *qualv(x,y)* ”x is a quality value of the quality y”

B16. *propins(x,y)* ”quality x is an instance of property y”

B17. *propvins(x,y)* ”quality value x is an instance of property value y”

B18. *MeasSys(x)* ”measurement system x”

B19. *meas(x,y)* ”x is the measurement system of property y”

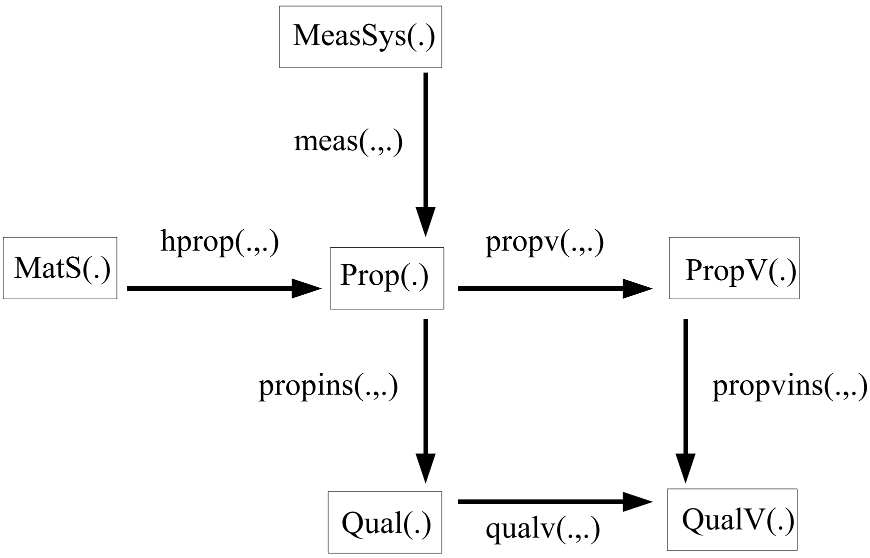


Figure 5.7: Context of Properties and Qualities

5.8.3 Occupied Space

B20. *extsp(x,y)* ”y is the extension-space of x”

B21. *occ(x,y)* ”x occupies the space entity y”

5.8.4 Amount of Substrate

B22. *PSubst(x)* "x is a presential amount of substrate"

B23. *PSol(x)* "x is an amount of solid substrate"

B24. *PGas(x)* "x is an amount of gaseous substrate"

B25. *PLiq(x)* "x is an amount of liquid substrate"

B26. *consist(x,y)* "x consists of the presential amount of substrate y"

B27. *MC(x)* "x is material connected"

5.9 Definitions

5.9.1 Standard Definitions

D71. $mppart(x,y) \Leftrightarrow mpart(x,y) \wedge x \neq y$ "x is a material proper part of y"

D72. $mov(x,y) \Leftrightarrow \exists z\ (mpart(z,x) \wedge mpart(z,y))$ "material overlap of material entities"

D73. $aggr_n(x_1, ..., x_n) = x \Leftrightarrow \forall x`(mov(x`, x) \leftrightarrow \bigvee_{i=1}^{n} mov(x`, x_i))$

"material sum(=aggregate) of x_1,...,x_n"

D74. $mintersect_n(x_1, ..., x_n) = x \Leftrightarrow \forall x`(mpart(x`, x) \leftrightarrow \bigwedge_{i=1}^{n} mpart(x`, x_i))$

"material intersection of x_1,...,x_n"

D75. $mrelcompl_n(x_1, ..., x_n) = x \Leftrightarrow \forall x`(mpart(x`, x) \leftrightarrow \bigwedge_{i=1}^{n-1} \neg mpart(x`,x_i) \wedge mpart(x`,x_n))$

"material relative complement of x_n and x_1,...,x_{n-1}"

5.9.2 Amount of Substrate

The following definitions are dealing with the state of aggregation of a certain material structure. We will define the classical three cases and a material structure which consists of different presential amounts of substrates like a natatorium or a glass of water.

D76. $Body(x) \Leftrightarrow MatS(x) \wedge \forall y\ (consist(x,y) \rightarrow PSol(y))$

"x is a solid body"

D77. $Gas(x) \Leftrightarrow MatS(x) \wedge \forall y\ (consist(x,y) \rightarrow PGas(y))$

"x is a gaseous material structure"

D78. $Fluid(x) \Leftrightarrow MatS(x) \wedge \forall y\ (consist(x,y) \rightarrow PLiq(y))$

"x is a fluid"

D79. $Mix(x) \Leftrightarrow MatS(x) \wedge \exists yz(mpart(y,x) \wedge mpart(z,x) \wedge ((Body(y) \wedge Gas(z)) \vee (Gas(y) \wedge Fluid(z)) \vee (Fluid(y) \wedge Body(z))))$

"x consists of different presential amounts of substrate"

5.9.3 Material Boundaries

D80. $2DMB(x) \Leftrightarrow \exists y\ (MatS(y) \wedge mb(x,y))$

"x is a 2-dimensional material boundary (surface region)"

D81. $1DMB(x) \Leftrightarrow \exists y\ (2DMB(y) \wedge mb(x,y))$ "

x is a 1-dimensional material boundary (line region)"

D82. $0DMB(x) \Leftrightarrow \exists y\ (1DMB(y) \wedge mb(x,y))$

"x is a 0-dimensional material boundary (point region)"

D83. $MB(x) \Leftrightarrow \exists y\ mb(x,y)$

"x is a material boundary"

D84. $ME(x) \Leftrightarrow MB(x) \vee MatS(x)$

"x is a material entity"

D85. $maxmb(x,y) \Leftrightarrow mb(x,y) \wedge \forall x'\ (mb(x',y) \rightarrow mpart(x',x))$

"x is maximal material boundary of y"

D86. $MaxMB(x)=y \Leftrightarrow maxmb(y,x)$

"maximal material boundary function"

D87. $2dmb(x,y) \Leftrightarrow MatS(y) \wedge mb(x,y)$

"x is a 2-dimensional material boundary (surface region) of y"

D88. $1dmb(x,y) \Leftrightarrow M2DB(y) \wedge mb(x,y)$

"x is a 1-dimensional material boundary (line region) of y"

D89. $0dmb(x,y) \Leftrightarrow M1DB(y) \wedge mb(x,y)$

"x is a 1-dimensional material boundary (point region) of y"

5.9.4 Properties and Qualities

In this subsection we want to generalize some binary basic relation to relations with n arguments. We will need these extended versions to define homogeneous material structures in an elegant way.

D90. $hprop(x,y_1,\ldots,y_n) \Leftrightarrow (MatS(x) \vee MB(x)) \wedge (\bigwedge_{i=1}^{n} Prop(y_i) \wedge hprop(x,y_i)) \wedge (\bigwedge_{1\leq i<j\leq n} y_i \neq y_j)$

"material entity x has property $y_1,\ldots,y_n$"

D91. $propins(x_1,\ldots,x_n,y_1,\ldots,y_n) \Leftrightarrow (\bigwedge_{i=1}^{n} Qual(x_i) \wedge Prop(y_i) \wedge propins(x_i,y_i))$

"qualities $x_1,\ldots,x_n$ are instances of the properties $y_1,\ldots,y_n$"

D92. $propvins(x_1,\ldots,x_n,y_1,\ldots,y_n) \Leftrightarrow (\bigwedge_{i=1}^{n} QualV(x_i) \wedge PropV(y_i) \wedge propvins(x_i,y_i))$

"quality values $x_1,\ldots,x_n$ are instances of the property values $y_1,\ldots,y_n$"

D93. $hqual(x,y_1,\ldots,y_n) \Leftrightarrow \exists z_1\ldots z_n(hprop(x,z_1,\ldots,z_n) \wedge propins(y_1,\ldots,y_n,z_1,\ldots,z_n))$

"material entity x has quality $y_1,\ldots,y_n$"

D94. $hindqual(x,y_1,\ldots,y_n) \Leftrightarrow MatS(x) \wedge \exists x`(maxmb(x`,x) \wedge hqual(x`,y_1,\ldots,y_n))$

"maximal material boundary of x induces the qualities $y_1,\ldots,y_n$ to x"

D95. $qualv(x_1,\ldots,x_n,y_1,\ldots,y_n) \Leftrightarrow (\bigwedge_{i=1}^{n} QualV(x_i) \wedge Qual(y_i) \wedge qualv(x_i,y_i))$

"$x_1,\ldots,x_n$ are quality values of the qualities $y_1,\ldots,y_n$"

D96. $propv(x_1,\ldots,x_n,y_1,\ldots,y_n) \Leftrightarrow (\bigwedge_{i=1}^{n} PropV(x_i) \wedge Prop(y_i) \wedge propv(x_i,y_i))$

"$x_1,\ldots,x_n$ are property values of the properties $y_1,\ldots,y_n$"

D97. $eqpropv(x_1,\ldots,x_n,y_1,\ldots,y_n) \Leftrightarrow \exists x_1`\ldots x_n`y_1`\ldots y_n`z_1\ldots z_n(qualv(x_1`,\ldots,x_n`,x_1,\ldots,x_n) \wedge qualv(y_1`,\ldots,y_n`,y_1,\ldots,y_n) \wedge propvins(x_1`,\ldots,x_n`,z_1,\ldots,z_n) \wedge propvins(y_1`,\ldots,y_n`,z_1,\ldots,z_n)$

"quality values of the qualities $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are instances of the same property values"

D98. $eqpropv_{z_1,\ldots,z_n}(x_1,\ldots,x_n,y_1,\ldots,y_n) \Leftrightarrow \exists x_1`\ldots x_n`y_1`\ldots y_n`(qualv(x_1`,\ldots,x_n`,x_1,\ldots,x_n) \wedge qualv(y_1`,\ldots,y_n`,y_1,\ldots,y_n) \wedge propvins(x_1`,\ldots,x_n`,z_1,\ldots,z_n) \wedge propvins(y_1`,\ldots,y_n`,z_1,\ldots,z_n)$

"quality values of the qualities $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are instances of the property values $z_1,\ldots,z_n$"

5.9.5 Homogeneity

Homogeneous Material Entities

A material structure x is homogeneous with respect to certain properties $y_1,\ldots,y_n$ if and only if every material part x‘ has the properties $y_1,\ldots,y_n$ with the same values[11]. Solid bodies, gaseous material structures and fluids are examples of homogeneous material entities with respect to the property "state of aggregation". A bicolored flag is an example of an inhomogeneous material structure with respect to the property "color".

Note that there are some properties of material structures like "volume" which cannot be transfered to their material parts. That means it does not make sense to talk about homogeneous material structures with respect to the property "volume" because the property values of this property vary from material part to material part.

D99. $HomMatS_{y_1,\ldots,y_n}(x) \Leftrightarrow hprop(x,y_1,\ldots,y_n) \wedge \exists z_1\ldots z_n\ (propins(z_1,\ldots,z_n,y_1,\ldots,y_n) \wedge hqual(x,z_1,\ldots,z_n) \wedge \forall\ x`\ (mpart(x`,x) \rightarrow hprop(x`,y_1,\ldots,y_n) \wedge \exists z_1`\ldots z_n`$
$(propins(z_1`,\ldots,z_n`,y_1,\ldots,y_n) \wedge hqual(x`,z_1`,\ldots,z_n`) \wedge eqpropv(z_1,\ldots,z_n,z_1`,\ldots,z_n`))))$

[11]Remember that all material parts of a certain material entity belong to same granularity level (corresponding sets).

"x is a homogeneous material structure with regard to the properties y_1,...y_n"

The following definition D100 of a homogeneous material structure provides a exact information about the instantiated property values in contrast to definition D99. Both definitions are equivalent. That means x is a homogeneous material structure in the sense of D99 if and only if x is a homogeneous material structure in the sense of D100.

D100. $HomMatS_{y_1,\ldots,y_n,y_1\text{`},\ldots,y_n\text{`}}(x) \Leftrightarrow hprop(x,y_1,\ldots,y_n) \wedge \exists z_1\ldots z_n\ (propins(z_1,\ldots,z_n,y_1,\ldots,y_n) \wedge hqual(x,z_1,\ldots,z_n) \wedge \forall\ x\text{`}\ (mpart(x\text{`},x) \rightarrow hprop(x\text{`},y_1,\ldots,y_n) \wedge \exists z_1\text{`}\ldots z_n\text{`}\ (propins(z_1\text{`},\ldots,z_n\text{`},y_1,\ldots,y_n) \wedge hqual(x,z_1\text{`},\ldots,z_n\text{`}) \wedge eqpropv_{y_1\text{`},\ldots,y_n\text{`}}(z_1,\ldots,z_n,z_1\text{`},\ldots,z_n\text{`}))))$

"x is a homogeneous material structure with regard to the properties y_1,...y_n and its quality values are instances of the property values y_1',...y_n'"

Maximal Homogeneous Material Parts

Inhomogeneous material entities, which do not have continuous changes of certain properties, have homogeneous material parts[12]. Consider therefore the following definitions.

D101. $hommpart_{y_1,\ldots,y_n}(x,y) \Leftrightarrow mpart(x,y) \wedge HomMatS_{y_1,\ldots,y_n}(x)$

"x is a homogeneous material part of the material structure y"

D102. $hommpart_{y_1,\ldots,y_n,y_1\text{`},\ldots,y_n\text{`}}(x,y) \Leftrightarrow mpart(x,y) \wedge HomMatS_{y_1,\ldots,y_n,y_1\text{`},\ldots,y_n\text{`}}(x)$

"x is a homogeneous material part of the material structure y"

For reasons of completeness we will give two versions of a maximal homogeneous material part of a certain material structure. Note that only the second version has a secure existence if there is at least one homogeneous material part[13]. That is why we introduced two different kinds of definitions of a homogeneous material structure.

D103. $maxhommpart_{y_1,\ldots,y_n}(x,y) \Leftrightarrow hommpart_{y_1,\ldots,y_n}(x,y) \wedge \forall x\text{`}(hommpart_{y_1,\ldots,y_n}(x\text{`},y) \rightarrow mpart(x\text{`},x))$

"x is a maximal homogeneous material part of the material structure y"

[12]We will talk about continuous changes in subsection 5.9.11

[13]The german flag has no maximal homogeneous material part with respect to the property "color" (definition D103) but it has a maximal homogeneous material part with respect to the property "color" and the property value "red"(D104).

D104. $maxhommpart_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(x,y) \Leftrightarrow hommpart_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(x,y) \wedge \forall x`$ $(hommpart_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(x`,y) \rightarrow mpart(x`,x))$

"x is a maximal homogeneous material part of the material structure y"

Comparability and Relative Equality

The analyzed properties may differ for every single consideration. If we fix a set of properties, then we may define indistinguishability or relative equality of two material structures. A necessary condition for the indistinguishability of two material structures is the comparability of them. That means they have to have the determined properties[14].

D105. $comp_{y_1,\ldots,y_n}(x,x`) \Leftrightarrow hprop(x,y_1,\ldots,y_n) \wedge hprop(x,y_1,\ldots,y_n)$

"x and x‘ are comparable with respect to the properties $y_1,\ldots,y_n$"

We will introduce four different definitions of relative equality of two material entities. The first definition is dealing with homogeneous material structures. That means we consider properties of a certain homogeneous material structure which are transferable to its material parts.

D106. $releq1_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(x,x`) \Leftrightarrow HomMatS_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(x) \wedge HomMatS_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(x`)$

"x and x‘ are relative equal with respect to properties $y_1,\ldots,y_n$ and property values $y_1`,\ldots,y_n`$"

Relative equality in the sense of definition D106 is a very strong condition because the considered material structures have to be homogeneous. One may temper this condition by regarding to maximal homogeneous material parts. Consider therefor the following definition.

D107. $releq2_{y_1,\ldots,y_n}(x,y) \Leftrightarrow \forall x`y_1`\ldots y_n` \; (maxhommpart_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(x`,x) \rightarrow \exists y`$ $(maxhommpart_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(y`,y))) \wedge \forall y`y_1`\ldots y_n` \; (maxhommpart_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(y`,y) \rightarrow$ $\exists x` \; (maxhommpart_{y_1,\ldots,y_n,y_1`,\ldots,y_n`}(x`,x)))$

"x and y are relative equal with respect to maximal homogeneous material parts and the properties $y_1,\ldots,y_n$"

[14] In subsection 5.7.3 we explained that the perception of properties is linked to certain granularity levels ("living spaces"). That means the comparability of two material entities allows their consideration at the same time, even if they belong to different granularity levels.

The definition D108 includes properties of a certain material entity which are untransferable to its material parts. Note that the relative equality of two different material structures in the sense of definition D106 implies the relative equality of them in the sense of D108.

D108. *$releq3_{y_1,\ldots,y_n}(x,y) \Leftrightarrow \exists z_1 \ldots z_n z_1` \ldots z_n`$ ($propins(z_1,\ldots,z_n,y_1,\ldots,y_n) \wedge$ $propins(z_1`,\ldots,z_n`,y_1,\ldots,y_n) \wedge hqual(x,z_1,\ldots,z_n) \wedge hqual(y,z_1`,\ldots,z_n`) \wedge$ $eqpropv(z_1,\ldots,z_n,z_1`,\ldots,z_n`)$)*

"x and y are relative equal with respect to the property values of their properties $y_1,\ldots,y_n$"

The last definition of relative equality of two material structures is dealing with qualities of their maximal material boundaries (so-called "induced qualities"). We will give some examples of relative equal material structures after the following definition.

D109. *$releq4_{y_1,\ldots,y_n}(x,y) \Leftrightarrow \exists z_1 \ldots z_n z_1` \ldots z_n`$ ($propins(z_1,\ldots,z_n,y_1,\ldots,y_n) \wedge$ $propins(z_1`,\ldots,z_n`,y_1,\ldots,y_n) \wedge hindqual(x,z_1,\ldots,z_n) \wedge hindqual(y,z_1`,\ldots,z_n`) \wedge$ $eqpropv(z_1,\ldots,z_n,z_1`,\ldots,z_n`)$*

"x and y are relative equal with respect to the property values of the properties $y_1,\ldots,y_n$ of their maximal material boundaries"

Figures

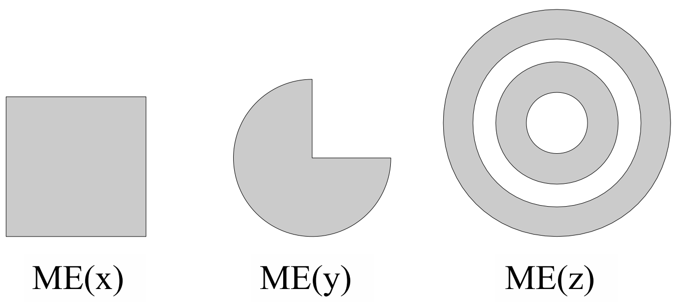


Figure 5.8: Relative Equality 1 (Color)

The figure above shows three different material entities which are relative equal (definition D106) with respect to the property "color". These entities are examples for the definitions D107 and D108, too. The next three material entities have three different maximal homogeneous material parts with respect to the property ”color”. Note that these entities do not fulfil the conditions of D106.

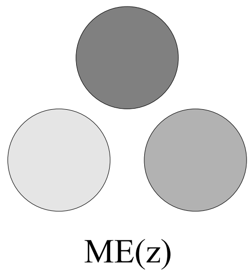


Figure 5.9: Relative Equality 2 (Color)

The following three material entities are equal (definition D108) with respect to the property ”surface area”. This equality cannot be expressed by definitions D106 and D107 because they are dealing with material parts.

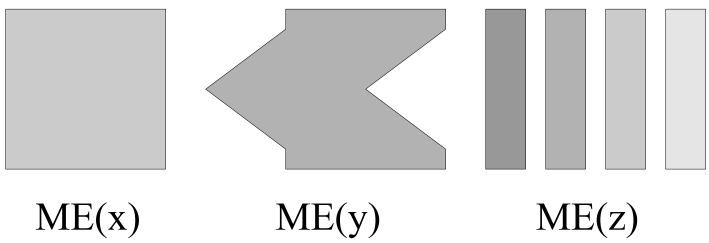


Figure 5.10: Relative Equality 3 (Surface Area)

The last figure exemplifies equal induced qualities (definition D109). All entities have the same shape but they differ in their volume and position. Note that we implicitly assume that the shape does not change by a translation, rotation or a concentric dilation or compression.

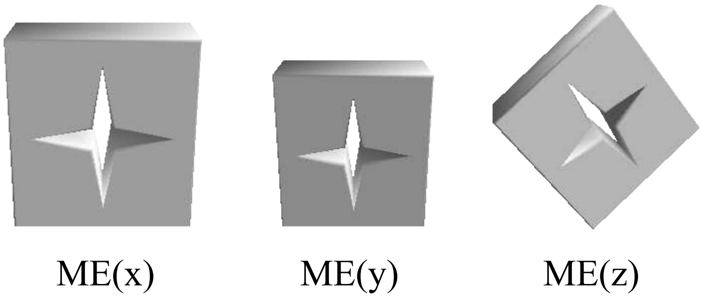


Figure 5.11: Relative Equality 4 (Shape)

5.9.6 Occupied Space

The maximal occupied space of every material entity depends on granularity. The finer the granularity level is the more detailed is the occupied space. We will assume the existence of a maximal occupied space for every material entity. The uniqueness will be shown in theorem T72[15].

D110. *maxocc(x,y)* $\Leftrightarrow$ *occ(x,y)* $\wedge$ $\forall$*y‘(occ(x,y‘)* $\rightarrow$ *spart(y‘,y))*

"y is the maximal occupied space of x"

D111. *MaxOcc(x)=y* $\Leftrightarrow$ *maxocc(x,y)* "maximal occupy-function"

Two material structures are "g-equal" if and only if they have the same maximal occupied space. In subsection 5.7.4 we showed that g-equality of two material structures x and y is not sufficient for the equality of x and y.

D112. $x{=}_g y \Leftrightarrow MaxOcc(x_g){=}MaxOcc(y_g)$

"maximal occupied spaces of x and y are equal at the granularity level g"

5.9.7 Material Connectedness

We want to distinguish between two-, one- and zero-dimensional material connectedness just like in case of spatial entities. The kind of material connectedness depends on granularity but not the material connectedness per se. Granularity plays a role because the occupied space is included in the definitions. One may say that it is not possible that two

[15]That means the maximal occupy relation is in fact a function.

material structures are material connected by a material point or line because they do not consist of an amount of substrate. That is right but this is only a kind of abstraction which is common practice in daily life. Imagine therefore two iron-globes which are weld together at a "point". Of course they are not material connected by a point but it looks like a point (at a certain level of granularity).

D113. *2DMC(x)* $\Leftrightarrow \exists y$ *(MatS(x)* $\wedge$ *maxocc(x,y)* $\wedge$ *2DC(y)* $\wedge$ *MC(x))*

"x is 2-dimensional material connected"

D114. *1DMC(x)* $\Leftrightarrow \exists y$ *(MatS(x)* $\wedge$ *maxocc(x,y)* $\wedge$ *1DC(y)* $\wedge$ *MC(x))*

"x is 1-dimensional material connected"

D115. *0DMC(x)* $\Leftrightarrow \exists y$ *(MatS(x)* $\wedge$ *maxocc(x,y)* $\wedge$ *0DC(y)* $\wedge$ *MC(x))*

"x is 0-dimensional material connected"

D116. *mc(x,y)* $\Leftrightarrow$ *MC(aggr(x,y))* $\wedge \neg$*mov(x,y)*

"x and y are material connected"

5.9.8 Separateness and Contact

D117. *sep(x,y)* $\Leftrightarrow \neg$*sov(MaxOcc(x),MaxOcc(y))* $\wedge \neg$*c(MaxOcc(x),MaxOcc(y))*

"x and y are separated"

D118. *con(x,y)* $\Leftrightarrow \neg$*sov(MaxOcc(x),MaxOcc(y))* $\wedge \neg$*mc(x,y)* $\wedge \exists$*x‘y‘ (mb(x‘,x)* $\wedge$ *mb(y‘,y)* $\wedge$ *scoinc(MaxOcc(x‘),MaxOcc(y‘)))*

"x and y are in contact"

We may distinguish now three different cases of relation between two non-overlapping material structures x and y, namely 1. x and y are separated 2. x and y are in contact and 3. x and y are material connected.

5.9.9 Kinds of Material Boundaries

We want to introduce two different kinds of material boundaries, namely *natural* and *fictitious material boundaries*. Natural material boundaries are divided into three different classes. A natural material boundary x of a certain material structure y with respect to the properties $y_1,...,y_n$ satisfies the following terms: 1. There is another non-material overlapping material structure u which is material connected or at least in contact with y 2. The material structure u has a material boundary v so that the maximal occupied spatial boundaries of v and x are coincident[16]; 3. The material structures u and y (or a material part y‘ of y with material boundary x) are distinguishable by the properties $y_1,...,y_n$.

If u and y are in contact like two homogeneous material cubes placed side by side, then we want to call the material boundaries at the touching area of them natural, even if they are indistinguishable in terms of their properties[17]. If u and y are indistinguishable and material connected, then we will call their material boundaries fictitious.

D119. $natmb_{con,y_1,\ldots,y_n}(x,y) \Leftrightarrow mb(x,y) \wedge \exists uv\ (con(u,y) \wedge mb(v,u) \wedge scoinc(MaxOcc(v),MaxOcc(x)) \wedge \exists y‘\ (mpart(y‘,y) \wedge mb(x,y‘) \wedge \exists y_1‘\ldots y_n‘ y_1“\ldots y_n“\ (\bigwedge_{i=1}^{n} y_i‘ \neq y_i“ \wedge HomMatS_{y_1,\ldots,y_n,y_1‘,\ldots,y_n‘}(y‘) \wedge HomMatS_{y_1,\ldots,y_n,y_1“,\ldots,y_n“}(u))))$

"x is a natural material boundary of y with respect to the properties $y_1,...,y_n$"

D120. $natmb_{con,noprop}(x,y) \Leftrightarrow mb(x,y) \wedge \exists uv\ (con(u,y) \wedge mb(v,u) \wedge scoinc(MaxOcc(v),MaxOcc(x)) \wedge \forall y_1\ldots y_n\ \neg natmb_{con,y_1,\ldots,y_n}(x,y)$

"x is a natural material boundary of y with respect to the properties $y_1,...,y_n$"

D121. $natmb_{mc,y_1,\ldots,y_n}(x,y) \Leftrightarrow \neg natmb_{con,y_1,\ldots,y_n}(x,y) \wedge mb(x,y) \wedge \exists uv\ (mc(u,y) \wedge mb(v,u) \wedge scoinc(MaxOcc(v),MaxOcc(x)) \wedge \exists y‘\ (mpart(y‘,y) \wedge mb(x,y‘) \wedge \exists y_1‘\ldots y_n‘ y_1“\ldots y_n“\ (\bigwedge_{i=1}^{n} y_i‘ \neq y_i“ \wedge HomMatS_{y_1,\ldots,y_n,y_1‘,\ldots,y_n‘}(y‘) \wedge HomMatS_{y_1,\ldots,y_n,y_1“,\ldots,y_n“}(u))))$

"x is a theoretical material boundary of y with respect to the properties $y_1,...,y_n$"

D122. $fictmb_{mc,noprop}(x,y) \Leftrightarrow mb(x,y) \wedge \exists uv\ (mc(u,y) \wedge mb(v,u) \wedge scoinc(MaxOcc(v),MaxOcc(x)) \wedge \forall y_1\ldots y_n\ \neg natmb_{mc,y_1,\ldots,y_n}(x,y))$

"x is a fictitious material boundary of y "

[16] This condition is necessary to guarantee that the material structures y and u are in contact (or material connected) "at" the material boundary x.

[17] This is only a convention. One may introduce another denotation.

Examples

The following figure illustrates the Leonardo-puzzle which was mentioned in the preliminary of this chapter.

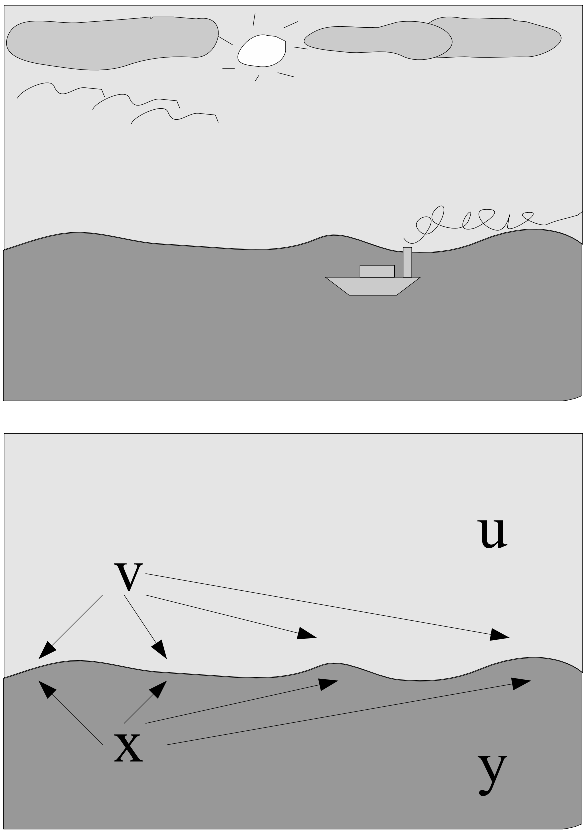


Figure 5.12: Natural Material Boundary D119 (Leonardo-puzzle)

The air is represented by the material structure u and the water is represented by the material structure y. The water and the air are in contact because there are material boundaries x of y and v of u which occupy coincident spatial boundaries. The material boundaries of the air and the water are even natural material boundaries (definition D119), because the air and the water are not material connected and they are distinguishable by the property "state of aggregation". What is "interposed" between the two natural boundaries are two coinciding spatial boundaries which do not occupy any three-dimensional space.

The following figure shows two material cubes with the same properties (identical in construction). The material boundaries at the touching area are examples of natural material boundaries in the sense of definition D120 because the cubes are in contact and furthermore indistinguishable with respect to their properties.

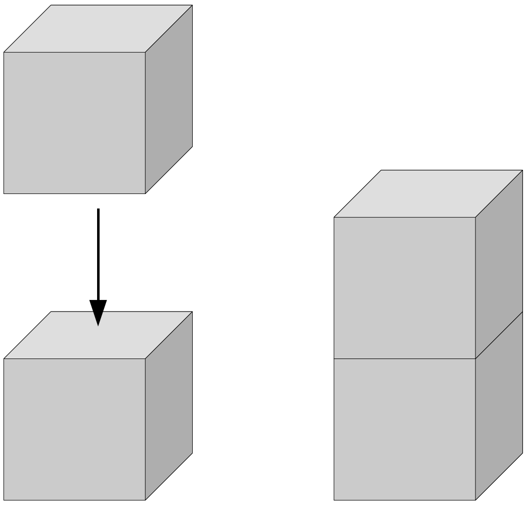


Figure 5.13: Natural Material Boundary D120 (Material Cubes)

The following figure shows a bar magnet. The positive charged side is colored in light grey and the negative charged side is colored in grey.

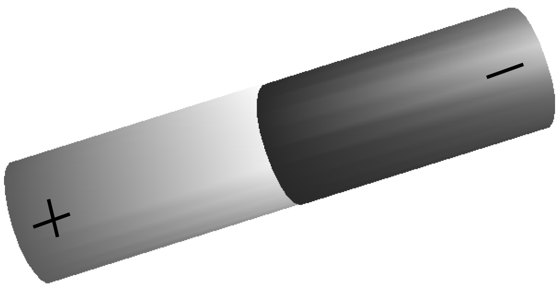


Figure 5.14: Natural Material Boundary D121 (Bar Magnet)

The material boundaries of the positive and the negative charged side at the "touching area" are natural material boundaries in the sense of definition D121. The reason of this is the material connectedness of the bar magnet and furthermore the distinguishability by the properties "color" or "charging".

The last kind of material boundaries are fictitious material boundaries. Imagine therefore a block of iron. Now we want to consider a material inner part of the block which looks like a three-dimensional eight.

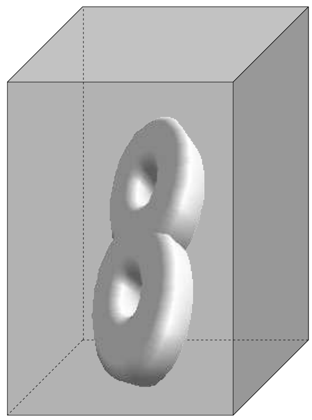


Figure 5.15: Fictitious Material Boundary (Material Eight)

The material boundary of the three-dimensional eight is fictitious because there are no properties which may distinguish the eight of the rest of the iron block and in contrast to figure 5.13 we have material connectedness of the eight and its environment.

Maximal Material Boundaries

D123. $maxnatmb_{con,y_1,\ldots,y_n}(x,y) \Leftrightarrow mb(x,y) \wedge \forall x`\ (natmb_{con,y_1,\ldots,y_n}(x`,y) \leftrightarrow mpart(x`,x))$

"x is a maximal natural material boundary of y"

D124. $maxnatmb_{con,noprop}(x,y) \Leftrightarrow natmb_{con,noprop}(x,y) \wedge \forall x`\ (natmb_{con,noprop}(x`,y) \rightarrow mpart(x`,x))$

"x is a maximal natural material boundary of y"

D125. $maxnatmb_{mc,y_1,\ldots,y_n}(x,y) \Leftrightarrow mb(x,y) \wedge \forall x`\ (natmb_{mc,y_1,\ldots,y_n}(x`,y) \leftrightarrow mpart(x`,x))$

"x is a maximal natural material boundary of y"

D126. $maxfictmb_{mc,noprop}(x,y) \Leftrightarrow fictmb_{mc,noprop}(x,y) \wedge \forall x`\ (fictmb_{mc,noprop}(x`,y) \rightarrow mpart(x`,x))$

"x is a maximal fictitious material boundary of y"

D127. $MaxNatMB_{con,y_1,\ldots,y_n}(y)=x \Leftrightarrow maxnatmb_{con,y_1,\ldots,y_n}(x,y)$

"maximal natural material boundary function"

D128. $MaxNatMB_{con,noprop}(y)=x \Leftrightarrow maxnatmb_{con,noprop}(x,y)$

"maximal natural material boundary function"

D129. $MaxNatMB_{mc,y_1,\ldots,y_n}(y)=x \Leftrightarrow maxnatmb_{mc,y_1,\ldots,y_n}(x,y)$

"maximal natural material boundary function"

D130. $MaxFictMB_{mc,noprop}(y)=x \Leftrightarrow maxfictmb_{mc,noprop}(x,y)$

"maximal fictitious material boundary function"

Some Remarks

Note that the definitions D123 and D125 are not defined in the usual way. The defined entities are only material boundaries and not natural material boundaries in the sense of the definitions D119 and D121. If we claim that they have to be natural, then we may have no maximal natural material boundaries even if there are natural material boundaries. Consider therefore the following rubber-example.

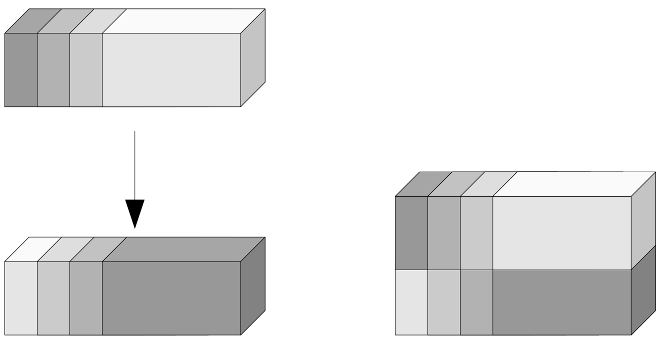


Figure 5.16: Rubber-Example

The figure 5.16 shows two different colored rubbers lying upon each other. We may find natural material boundaries (definition D119) with respect to the property "color" but we cannot find a maximal natural material boundary which has all natural material boundaries as material parts. The reason for this is that we cannot find a homogeneous material structure with respect to the property "color". Nevertheless, we may define a maximal variant if we reject the naturalness (definition D123).

The maximal material boundary of a certain material structure may consist of different kinds of material boundaries. Consider three bricks lying upon each other. Two of them are identical in construction. That means they cannot be distinguished by a property.

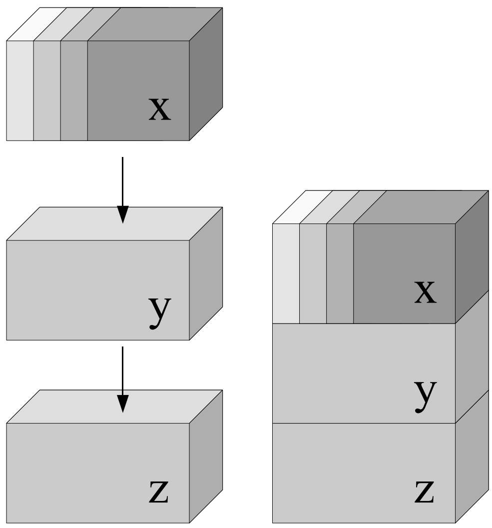


Figure 5.17: Brick-Example

The lower side of the brick y is a natural material boundary in the sense of definition D120 and the upper side is an example of definition D119. Hence the maximal material boundary of y consists of different kinds of material boundaries. With the help of the four different kinds of material boundaries we may introduce new terms for material structures.

5.9.10 Selected Solid Material Structures

We want to classify solid material structures with the help of two criteria: 1. Material connected with its environment[18] or contact between the material structure and its environment and 2. Distinguishable or indistinguishable (with respect to certain properties) from its environment.

All considered material structures are material connected. Note that this is not a complete classification of solid material structures but rather a first overview. Both first definitions are the most important. Almost all everyday objects like a car, a house or a computer monitor are "material objects" in the sense of definition D131. They are not material connected with its environment and furthermore distinguishable by at least one property like "color" or "material consistency". The second kind of solid material structures are called "material-part-object" because they are material connected with at least one part of its environment. Examples are the wheel of a car, the chimney of a house or the display of a monitor.

D131. $MatOb(x) \Leftrightarrow Body(x) \wedge MC(x) \wedge \exists yy_1\ (maxmb(y,x) \wedge maxnatmb_{con,y_1}(y,x))$

"x is a material object"

D132. $MatPartOb(x) \Leftrightarrow Body(x) \wedge MC(x) \wedge \exists yy_1\ (mb(y,x) \wedge maxnatmb_{mc,y_1}(y,x))$

"x is a material-part-object"

The following kind of solid material structures is indistinguishable with at least one part of its environment. The two cubes in figure 5.13 are examples of a "indistinguishable material object". Two human bones which are in contact (touching each other) are examples, too. Note that indistinguishable material objects in the sense of definition D133 are very rare because they have to be in contact with an object which is identical in construction.

D133. $IndMatOb(x) \Leftrightarrow Body(x) \wedge MC(x) \wedge \exists y\ (mb(y,x) \wedge maxnatmb_{con,noprop}(y,x))$

"x is a indistinguishable material object"

The last kind are "fictitious material objects". They are material connected with at least one part of its environment which cannot be distinguished from it. The three-dimensional

[18] The term "environment" is here used in the sense of a surrounding material structure.

eight in figure 5.15 is an example of this class of material structures. Other examples are the half wheel of a car or the half display of a monitor.

D134. $FictMatOb(x) \Leftrightarrow Body(x) \wedge MC(x) \wedge \exists y\ (mb(y,x) \wedge maxfictmb_{mc,noprop}(y,x))$

"x is a fictitious material object"

5.9.11 Some Remarks about Continuous Changes

Inhomogeneous material structures are divided into two classes, namely material structures which are compositions of homogeneous material structures and material structures with an continuous change of a certain property. Consider therefor the following figures.

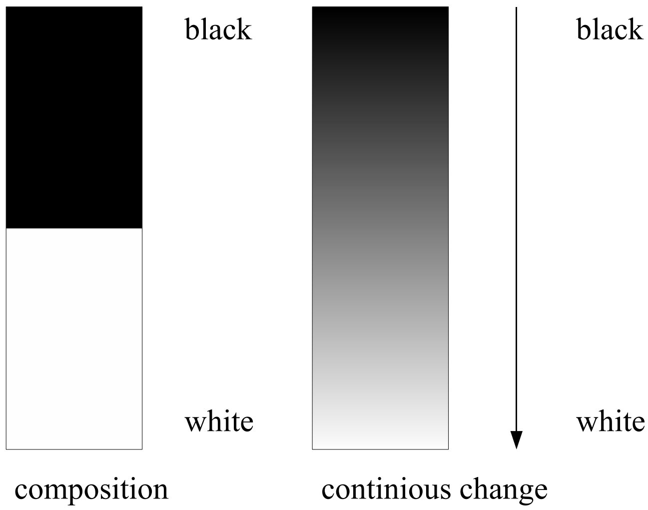


Figure 5.18: Continuous Change

The second figure is not a material aggregate of homogeneous material parts. Every material part of it is inhomogeneous with respect to the property "color". In the following we want to examine several characteristics of continuous changes. For reasons of presentability we will consider the property "color". Furthermore we will assume that the measurement system of it is represented by the standard closed interval $[0,1] \subset \mathbf{R}$. We identify the color white with the real number "0" and the color black with the real number "1"[19].

[19] One may generalize the standard closed interval to a triangle. Every corner represents one of the primary color blue, green or red.

Direction

A colored surface has a vertical (width) and a horizontal extension (hight). In the right example in figure 5.18 we observe that there is a vertical continuous change of the property "color" but no horizontal continuous change. If we consider a certain "direction-line" we may define a corresponding function. Consider therefore the following graphs.

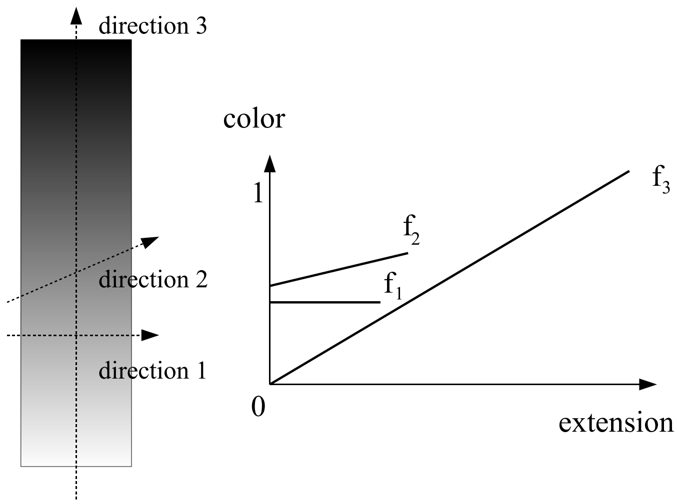


Figure 5.19: Direction

Rate of Change

In the diagram above we observe different linear functions. The slope of these linear function may be interpreted as "rate of change". Note that a certain homogeneous material structure has a constant rate of change in every "direction-line". The following four figures exhibit a continuous change of the property "color" in their vertical extension. The difference between their changes are the values of their rate of changes. The left figure has the lowest slope and the right one the highest.

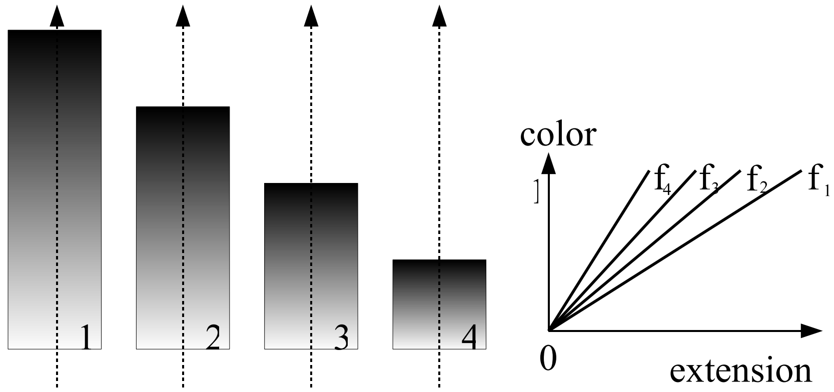


Figure 5.20: Rate of Change

Continuity and Monotony

With the help of the mathematical concepts "continuity" and "monotony" we may specify the notion of a continuous change. A colored surface has a continuous change (in a certain direction) if and only if its corresponding function is continuous and monotonic (compare figure 5.20). The following material entity has a corresponding discontinuous and unmonotonic function.

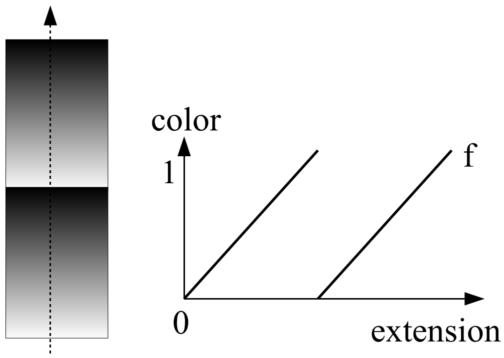


Figure 5.21: Continuity and Monotony

Final Remark

We considered a continuous change of the property "color". A car ride from Leipzig to Berlin or the age of a certain person are continuous changes, too. The location of the car and the age of the person may be interpreted as functions of time. It is future work to integrate several kinds of continuous changes in our axiomatization.

5.10 Axioms

5.10.1 Material Parts

The following axioms show that the material part relation is a partial ordering like its spatial analog. Keep in mind that they are dealing with different ontological categories.

A44. $\forall x\ (ME(x) \rightarrow mpart(x,x))$ "reflexivity of material part"

A45. $\forall xy\ (ME(x) \wedge ME(y) \wedge mpart(x,y) \wedge mpart(y,x) \rightarrow x{=}y)$

"antisymmetry of material part"

A46. $\forall xyz\ (ME(x) \wedge ME(y) \wedge ME(z) \wedge mpart(x,y) \wedge mpart(y,z) \rightarrow mpart(x,z))$

"transitivity of material part"

A47. $\forall xy\ (mpart(x,y) \rightarrow ME(x) \wedge ME(y))$ "range restriction"

A48. $\forall xy\ (\neg mpart(y,x) \rightarrow \exists z\ (mpart(z,y) \wedge \neg mov(z,x)))$

"strong supplementation principle (SSP)"

5.10.2 Occupied Space

A49. $\forall xy\ (occ(x,y) \rightarrow (MatS(x) \wedge SReg(y)) \vee (M2DB(x) \wedge 2DB(y)) \vee (M1DB(x) \wedge 1DB(y)) \vee (M0DB(x) \wedge 0DB(y)))$

"range restriction"

A50. $\forall x\ (ME(x) \rightarrow \exists y\ (maxocc(x,y)))$

"existence of a maximal occupied space entity"

5.10.3 Material Boundaries

Every material structure has a material boundary. This axiom cannot be generalized to lower-dimensional material entities (e.g. surface of a bowl). A conditional existence of maximal material boundaries can be claimed for material entities in general.

A51. $\forall xy\ (mb(x,y) \rightarrow ME(x) \wedge ME(y))$ "range restriction"

A52. $\forall xy\ (mb(x,y) \rightarrow \forall z\ (mpart(z,x) \rightarrow mb(z,y))$

"material parts of boundaries are boundaries"

Existence of Material Boundaries

A53. $\forall x\ (MatS(x) \rightarrow \exists y\ mb(y,x))$ "existence of a material boundary"

A54. $\forall xy\ (mb(x,y) \rightarrow \exists x`\ maxmb(x`,y))$

"conditional existence of a maximal material boundary"

A55. $\forall xyy_1 \ldots y_n\ (natmb_{con,y_1,\ldots,y_n}(x,y) \rightarrow \exists x`\ maxnatmb_{con,y_1,\ldots,y_n}(x`,y))$

"conditional existence of a maximal natural material boundary"

A56. $\forall xyy_1 \ldots y_n\ (natmb_{mc,y_1,\ldots,y_n}(x,y) \rightarrow \exists x`\ maxnatmb_{mc,y_1,\ldots,y_n}(x`,y))$

"conditional existence of a maximal natural material boundary"

A57. $\forall xy\ (natmb_{con,noprop}(x,y) \rightarrow \exists x`\ maxnatmb_{con,noprop}(x`,y))$

"conditional existence of a maximal natural material boundary"

A58. $\forall xy\ (fictmb_{mc,noprop}(x,y) \rightarrow \exists x`\ maxfictmb_{mc,noprop}(x`,y))$

"conditional existence of a maximal fictitious material boundary"

Embedding Postulation

The following axiom A59 claims that every material structure has either a material boundary which is natural (in the sense of definition D119) or may be embedded in such a material structure. This axiom is not very strong because the denial of it implies a homogeneous world and that is obviously not true.

A59. $\forall x\ (MatS(x) \rightarrow \exists x`yy_1\ (mpart(x,x`) \wedge maxmb(y,x`) \wedge maxnatmb_{con,y_1}(y,x`)))$

"embedding in material structures with a natural boundary"

5.10.4 Material Functions

In this subsection we want to postulate the conditions for the existence of material functions. Just like in case of their spatial analogs we will exclude material sums, intersections or relative complements of different-dimensional material entities.

A60. $\forall xy$ *((MatS(x)* $\wedge$ *MatS(y))* $\vee$ *(M2DB(x)* $\wedge$ *M2DB(y))* $\vee$ *(M1DB(x)* $\wedge$ *M1DB(y))* $\vee$ *(M0DB(x)* $\wedge$ *M0DB(y))* $\rightarrow$ $\exists$*aggr(x,y))*

"existence of material sum"

A61. $\forall xy$ *(mov(x,y)* $\rightarrow$ $\exists$*mintersect(x,y))* "existence of material intersection"

A62. $\forall xy$ *($\neg$mpart(y,x)* $\wedge$ *((MatS(x)* $\wedge$ *MatS(y))* $\vee$ *(M2DB(x)* $\wedge$ *M2DB(y))* $\vee$ *(M1DB(x)* $\wedge$ *M1DB(y))* $\vee$ *(M0DB(x)* $\wedge$ *M0DB(y))* $\rightarrow$ $\exists$*mrelcompl(x,y))*

"existence of material complement"

5.10.5 Interrelations between Material and Spatial Entities

The following axiom A63 stipulates that the Brentanoraum is not "empty". That means there is at least one material structure which occupies three-dimensional space. The axioms A64, A65 and A66 postulate the interrelations between material and spatial connectedness, material and spatial parts as well as material and spatial boundaries. Note that the converses are not true in general (compare subsections 5.3, 5.4 and 5.5)

A63. $\forall x$ *(Top(x)* $\rightarrow$ $\exists$*y,z,u (Top(y)* $\wedge$ *spart(x,y)* $\wedge$ *spart(u,y)* $\wedge$ *MatS(z)* $\wedge$ *MaxOcc(z,u))*

"existence of an extension with 'filled' parts"

Basic Relations

A64. $\forall x$ *(mc(x)* $\rightarrow$ *c(MaxOcc(x))*

"material connectedness implies spatial connectedness"

A65. $\forall xy$ *(mpart(x,y)* $\rightarrow$ *spart(MaxOcc(x),MaxOcc(y)))*

"material parts occupy spatial parts"

A66. $\forall xy\ (mb(x,y) \rightarrow sb(MaxOcc(x),MaxOcc(y)) \vee hypp(MaxOcc(x),MaxOcc(y)))$

"material boundaries occupy spatial boundaries respective hyper parts"

Remember that axiom A66 would be false if we exclude the second part of the disjunction. Consider therefore the "Mathias-example" in figure 5.3. This observation leads us directly to the following axiom, which establishes relations between extraordinary spatial entities and material entities.

Extraordinary Entities

A67. $\forall xy\ ((mb(x,y) \rightarrow hypp(MaxOcc(x),MaxOcc(y)) \wedge \neg sb(MaxOcc(x),MaxOcc(y))) \rightarrow \exists x`\ (mb(x`,y) \wedge ExOrd(MaxOcc(x`))))$

"existence of extraordinary entities"

The axiom A67 justificated the detailed analysis of extraordinary entities in the Brentanoraum.

Material Functions

The following axioms represent the compatibility of spatial and material operators. The maximal occupied space of a material sum of x and y is equal to spatial sum of the maximal occupied spaces of x and y. The same holds for the material intersection and complement.

A68. $\forall xy\ (\exists aggr(x,y) \rightarrow MaxOcc(aggr(x,y))= sum(MaxOcc(x),MaxOcc(y)))$

"compatibility of material and spatial sum"

A69. $\forall xy\ (\exists mintersect(x,y) \rightarrow MaxOcc(mintersect(x,y))= intersect(MaxOcc(x),MaxOcc(y)))$

"compatibility of material and spatial intersection"

A70. $\forall xy\ (\exists mrelcompl(x,y) \rightarrow MaxOcc(mrelcompl(x,y))= relcompl(MaxOcc(x),MaxOcc(y)))$

"compatibility of material and spatial complement"

5.10.6 Properties and Qualities

Range Restrictions

A71. $\forall xy\ (hprop(x,y) \rightarrow (ME(x)) \wedge Prop(y))$ "range restriction"

A72. $\forall xy\ (propv(x,y) \rightarrow (PropV(x)) \wedge Prop(y))$ "range restriction"

A73. $\forall xy\ (qualv(x,y) \rightarrow (QualV(x)) \wedge Qual(y))$ "range restriction"

A74. $\forall xy\ (propins(x,y) \rightarrow (Qual(x)) \wedge Prop(y))$ "range restriction"

A75. $\forall xy\ (propvins(x,y) \rightarrow (QualV(x)) \wedge PropV(y))$ "range restriction"

A76. $\forall xy\ (meas(x,y) \rightarrow (MeasSys(x)) \wedge Prop(y))$ "range restriction"

Dependency and Existence

A77. $\forall x\ (Prop(x) \rightarrow \exists y\ (MeasSys(y) \wedge meas(y,x))$

"properties have a measurement system"

A78. $\forall x\ (PropV(x) \rightarrow \exists y\ (Prop(y) \wedge propv(x,y))$

"property values belong to properties"

A79. $\forall x\ (Qual(x) \rightarrow \exists y\ (Prop(y) \wedge propins(x,y)) \wedge \exists z\ (ME(z) \wedge hqual(z,x)))$

"qualities are instances of properties and cannot exist in isolation"

The following axiom A80 assures that the diagram in figure 5.7 is "well-formed". That means the instantiated property value and the associated quality of a certain quality value belong to the same property.

A80. $\forall x\ (QualV(x) \rightarrow \exists yy'y''\ (Qual(y) \wedge Prop(y') \wedge PropV(y'') \wedge qualv(x,y) \wedge propins(y,y') \wedge propvins(x,y'') \wedge propv(y'',y')))$

"well-formed memberships"

A81. $\forall x\ (MeasSys(x) \rightarrow \exists y\ (Prop(y) \wedge meas(x,y)))$

"measurement systems belong to properties"

Material Entities

The following axiom A82 claims that material structures are not proptertyless. Furthermore they have an extension space (axiom A83). This intrinsic quality cannot be shared with other entities because of the principle of non-migration (axiom A84). A quality of a certain material structure inheres in this material structure.

A82. $\forall x\ (ME(x) \rightarrow \exists y\ hprop(x,y))$ "existence of properties"

A83. $\forall x\ (ME(x) \rightarrow \exists y\ extsp(y,x))$ "existence of extension space"

A84. $\forall xyz\ (ME(x) \wedge ME(y) \wedge hqual(x,z) \wedge hqual(y,z) \rightarrow x{=}y)$

"principle of non-migration"

5.10.7 Amount of Substrate

A85. $\forall xy\ (consist(x,y) \rightarrow (MatS(x)) \wedge PSubst(y))$ "range restriction"

A86. $\forall x\ (MatS(x) \rightarrow \exists y\ consist(x,y))$ "existence of a presential amount of substrate"

A87. $\forall x\ (PSubst(x) \rightarrow \exists y\ MatS(y) \wedge consist(y,x))$

"a presential amount of substrate depends on a material structure"

5.10.8 Sub-Categories and Disjointness

A88. $\forall x\ (PSol(x) \vee PGas(x) \vee PLiq(x) \rightarrow PSubst(x))$

"sub-categories of presential amounts of substrate"

A89. $\forall x\ (ME(x) \vee PSubst(x) \rightarrow Pres(x))$

"sub-categories of presentials"

The following axiom represents the pairwise disjointness between: MatS, M2DB, M1DB, M0DB, SReg, 2DB, 1DB, 0DB, Prop, PropV, Qual, QualV, MeasSys, PSol, PGas and PLiq. Because of the 120 conjunctive elements we will denote only the beginning.

A90. $\forall xy\ (\neg(MatS(x) \wedge M2DB(x)) \wedge \neg(MatS(x) \wedge M1DB(x)) \wedge ... \wedge \neg(MatS(x) \wedge PLiq(x)) \wedge \neg(M2DB(x) \wedge M1DB(x))...$

"disjointness"

5.11 Propositions

5.11.1 Identity Principles

The material part relation is a partial ordering on every class of material entities just like its spatial analog. Furthermore we claimed the strong supplementation principle for material entities in axiom A48. That is why we may deduce the correspondent identity principles for material entities. The proofs are analog to the theorems T1, T2, T3 and T4.

T60. $\forall xy\ (\forall z\ (mpart(z,x) \leftrightarrow mpart(z,y)) \leftrightarrow x=y)$ "1. identity principle"

T61. $\forall xy\ (\forall z\ (mpart(x,z) \leftrightarrow mpart(y,z)) \leftrightarrow x=y)$ "2. identity principle"

T62. $\forall xy\ (\exists z`\ (mppart(z`,x)) \wedge \forall z\ (mppart(z,x) \rightarrow mppart(z,y)) \rightarrow mpart(x,y))$

"material proper part principle"

T63. $\forall xy\ (\exists z`\ (mppart(z`,x) \vee (mppart(z`,y)) \rightarrow (x=y \leftrightarrow \forall z(mppart(z,x) \leftrightarrow mppart(z,y))))$

"3. identity principle"

5.11.2 Uniqueness of Material Functions

Standard Material Functions

The material sum, intersection and complement are functional. Analogous to the subsection above, we will give the following theorems without a proof because they are identical to the spatial case.

T64. $\forall xx`x_1...x_n(aggr(x_1,...,x_n) = x \wedge\ aggr(x_1,...,x_n) = x` \rightarrow x=x`)$

"uniqueness of material sum"

T65. $\forall xx`x_1...x_n(mintersect(x_1,...,x_n) = x \wedge$ *mintersect*$(x_1,...,x_n) = x` \rightarrow x{=}x`)$

"uniqueness of material intersection"

T66. $\forall xx`x_1...x_n(mrelcompl(x_1,...,x_n) = x \wedge$ *mrelcompl*$(x_1,...,x_n) = x` \rightarrow x{=}x`)$

"uniqueness of material relative complement"

Maximal Material Boundaries

T67. $\forall xyz$ *(maxmb(y,x)* $\wedge$ *maxmb(z,x)* $\rightarrow$ *y=z)*

"uniqueness of maximal material boundary"

Proof: assume maxmb(y,x) $\wedge$ maxmb(z,x); by D85(maximal material boundary) follows mb(y,x) $\wedge$ mb(z,x), therefore spart(y,z) $\wedge$ spart(z,y); hence **y=z** by antisymmetry of material part

T68. $\forall xyy_1 \ldots y_n z$ *(maxnatmb*$_{con,y_1,\ldots,y_n}$*(y,x)* $\wedge$ *maxnatmb*$_{con,y_1,\ldots,y_n}$*(z,x)* $\rightarrow$ *y=z)*

"uniqueness of maximal natural material boundary"

Proof: obvious, because both have the same material parts(1. identity principle); compare D123(maximal natural material boundary)

T69. $\forall xyz$ *(maxnatmb*$_{con,noprop}$*(y,x)* $\wedge$ *maxnatmb*$_{con,noprop}$*(z,x)* $\rightarrow$ *y=z)*

"uniqueness of maximal natural material boundary"

Proof: assume maxnatmb$_{\mathrm{con,noprop}}$(y,x) $\wedge$ maxnatmb$_{\mathrm{con,noprop}}$(z,x); by D124(maximal natural material boundary) follows natmb$_{\mathrm{con,noprop}}$(y,x) $\wedge$ natmb$_{\mathrm{con,noprop}}$(z,x), therefore mpart(y,z) $\wedge$ mpart(z,y); hence **y=z** by antisymmetry of material part

T70. $\forall xyy_1 \ldots y_n z$ *(maxnatmb*$_{mc,y_1,\ldots,y_n}$*(y,x)* $\wedge$ *maxnatmb*$_{mc,y_1,\ldots,y_n}$*(z,x)* $\rightarrow$ *y=z)*

"uniqueness of maximal natural material boundary"

Proof: obvious, because both have the same material parts(1. identity principle); compare D125(maximal natural material boundary)

T71. $\forall xyz\ (maxfictmb_{mc,noprop}(y,x) \wedge maxfictmb_{mc,noprop}(z,x) \rightarrow y{=}z)$

"uniqueness of maximal fictitious material boundary"

Proof: assume $\mathrm{maxfictmb}_{\mathrm{mc,noprop}}(\mathrm{y,x}) \wedge \mathrm{maxfictmb}_{\mathrm{mc,noprop}}(\mathrm{z,x})$; by D126(maximal fictitious material boundary) follows $\mathrm{fictmb}_{\mathrm{mc,noprop}}(\mathrm{y,x}) \wedge \mathrm{fictmb}_{\mathrm{mc,noprop}}(\mathrm{z,x})$, therefore mpart(y,z) ∧ mpart(z,y); hence **y=z** by antisymmetry of material part

Maximal Occupied Space

T72. $\forall xyz\ (maxocc(x,y) \wedge maxocc(x,z) \rightarrow y{=}z)$

"uniqueness of the maximal occupied space"

Proof: assume maxocc(x,y) ∧ maxocc(x,z); by D110(maximal occupied space) follows occ(x,y) ∧ occ(x,z), therefore spart(y,z) ∧ spart(z,y); hence **y=z** by antisymmetry of spatial part

5.11.3 Range Restrictions

T73. $\forall xy\ (mpart(x,y) \rightarrow (MatS(x) \wedge MatS(y)) \vee (M2DB(x) \wedge M2DB(y)) \vee (M1DB(x) \wedge M1DB(y)) \vee (M0DB(x) \wedge M0DB(y)))$

"range restriction"

Proof: assume mpart(x,y); with A47(range material part) we get ME(x) ∧ ME(y); by using A50(existence maximal occupied space) and A65(material parts occupy spatial parts) follows spart(MaxOcc(x),MaxOcc(y)); assume **MatS(x)**; with A4(equal dimension spatial part) and A49(range of occupy relation) follows **MatS(y)**; the same holds for material boundaries;

T74. $\forall xy\ mb(x,y) \wedge sb(MaxOcc(x),MaxOcc(y)) \rightarrow (M2DB(x) \wedge MatS(y)) \vee (M1DB(x) \wedge M2DB(y)) \vee (M0DB(x) \wedge M1DB(y))$

"range restriction"

Proof: assume mb(x,y) ∧ sb(MaxOcc(x),MaxOcc(y)); with A51(range material boundary) we get ME(x) ∧ ME(y); by using A50(existence maximal occupied space); assume **M2DB(x)**; with A15(range spatial part) and A49(range of occupy relation) follows **MatS(y)**; the same holds for the other cases

Chapter 6

Conclusion

6.1 Summary of Results

The aim of this thesis was the development of an axiomatic foundation of an ontology of spatial and material entities. The first step was an introduction and philosophical discussion of ontological views like realism, nominalism and conceptualism. Furthermore we illustrated our pluralistic approach which is expressed by the postulation of three kinds of universals in GFO, namely *immanent universals*, *conceptual structures* and *symbolic structures*. With the help of several ontological relations like instantiation, correlation and denotation we clarified the connection between these categories and a certain material entity.

In the third chapter we explained in detail the cognitively inadequateness of the real space $\mathbf{R}^3$ as a model of the surrounding space. The main problem is that contact between two similar objects cannot be explained. This problem is caused by the open/closed distinction in the real space. The standard model $\mathbf{R}^3$ represents a down-to-top approach. Extended three-dimensional space regions are interpreted as a set of (unexpanded) points. The philosopher Franz Brentano opposed the association of the surrounding space as a mathematical continuum and argued for a top-to-down approach which is closer to our cognition. The ideas of Brentano have been a source of inspiration for our theory of space. That is why we introduced the term "Brentanoraum $\mathbf{B}^3$".

In chapter four we introduced the Brentanoraum in detail. The core of our axiomatization is a *classical extensional mereology (CEM).* The universe of discourse of the Brentanoraum is divided into four classes, namely three-dimensional space regions and lower-dimensional surface, line and point regions. Space regions correspond to compact three-dimensional manifolds which are embeddable into $\mathbf{R}^3$. The most important subclass of space regions are so-called *topoids*. They are defined as spatial connected space regions. The importance of them is caused by the fact that almost all material objects occupy topoids. According to the 1. Brentanian Thesis we claimed that lower-dimensional entities (=spatial boundaries) cannot exist in isolation. A ordinary spatial boundary x is a spatial boundary of a certain higher-dimensional spatial entity y.

We introduced a number of definitions to enable a detailed classification of spatial entities. Beside the standard definitions of a mereological system we considered new concepts like "connected components" and "extraordinariness". We introduced three different versions of connected components and analyzed their interrelations, e.g. CC-inequality. A spatial entity is said to be extraordinary if it has two non-overlapping coincident spatial parts. An important subclass of extraordinary spatial entities is constituted "cross-entities". They usually appear if two spatial entities interpenetrate or cross (touch) each other.

Let us recapitulate our notion of mereotopological elementary equivalence. In simple words one may say that two spatial entities are elementary equivalent if and only if the same sentences (with respect to a certain signature) are true about them. That means two spatial entities belong to different equivalence classes if we find a formula φ which distinguish them. We showed for instance that the two-dimensional sphere and a torus are distinguishable with respect to the primitives of the Brentanoraum.

In the fifth chapter we extended the Brentanoraum to a spatial-material theory. A certain material structure is an individual that fulfils the following conditions: it is a presential, it is a bearer of qualities , it occupies space and it consists of an presential amount of substrate. The interrelation of spatial and material entities is given by the *occupy-relation*. Every material entity occupies a spatial entity but not necessarily vice versa. This ability is caused by the intrinsic quality to have a spatial extension, which is called the *extension space*. The determination of the maximal occupied space of a certain material structure depends on granularity, vagueness and context. Furthermore, we introduced the granularity-function and therefore the corresponding sets of material parts, material boundaries, qualities and their belonging values of a certain material structure. We motivated that these sets are also not invariant if we change the granularity level. That is why we fix the granularity level for every single consideration.

We defined several concepts of relative equality (with respect to certain properties $y_1,...,y_n$) between two material structures. These definitions provide a possibility to compare several material structures. In physical considerations it might be useful to classify objects with respect to their state of aggregation, density or intrinsic energy. A main focus was on material boundaries. We distinguished four kinds of them, namely *fictitious material boundaries* and three classes of *natural material boundaries*. Naturalness is given if there is a qualitative heterogeneity of the bounded entity and its environment (complement). In case of indistinguishability and material connectedness of a certain material structure and its complement we talk about fictitious material boundaries. This classification captures almost all perceptible material boundaries of material structures in the real world.

6.2 Comparison to [Smi, Var 2000]

There are other axiomatic foundations of an ontology of space and boundaries. In this section we want to compare our theory of boundaries with [Smi, Var 2000]. Smith and Varzi developed two complementary theories of boundaries with a common core[1]. The first kind of boundaries, so-called *bona fide boundaries* correspond to "natural" boundaries like the surface of the moon or the surface of a tennis ball. The term "natural" implies a qualitative heterogeneity of the bounded entity and its environment (complement). The second kind are *fiat boundaries*. Examples are national borders or property lines and inner boundaries of objects without any physical discontinuity or qualitative differentiation to its environment.

6.2.1 Bona Fide Boundaries

According to the explanations in this paper, bona fide boundaries are considered as boundaries of everyday objects like a car, a house or a human being. A material object x is "closed" if all boundaries of x are spatial parts of x and "open" if all bona fide boundaries are spatial parts of the complement of x.

In subsection 3.2.1 we discussed the problems of an open/closed distinction. One has to decide what kinds of objects are open and what are closed. This decision has to

[1] common core = standard mereological system + dependency thesis

exclude peculiar privileging. It turns out that two similar objects cannot be in contact because they are either closed or open. This is the strongest point of criticism of this approach because physical contact between two similar objects cannot be explained. Consider therefore the following citation:

> "Thus, contact between John and Mary is simply not possible if this is understood in terms of external connection. This is in agreement with physics and ordinary topology."(cited in [Smi, Var 2000] p.16)

The observable and perceptible contact between two kissing persons is no contact in the sense of this theory.

Another point of criticism is that bona fide boundaries are symmetric in the sense that if x is a bona fide boundary of y, then x is a bona fide boundary of the complement of y, too. That means that the natural boundary of the house is a natural boundary of the surrounding air, too. This interpretation is a intermixture of two terms, namely boundary in the sense of "part of the object" and boundary as "demarcating entity".

In our theory we introduced three different kinds of natural boundaries (compare definitions D119, D120, D121) which avoid the problems mentioned above. These definitions imply a qualitative heterogeneity of the bounded entity and its environment (complement), too. Nevertheless, it is possible to define contact between two similar and not similar objects (compare definition D118).

6.2.2 Fiat Boundaries

The theory of fiat boundaries leaves ordinary topology and follows the Brentanian idea of coincidence. Fiat boundaries require an additional act of thought because they do not introduce any physical discontinuity.

Fiat boundaries correspond to our fictitious boundaries (compare definition D122). Note that we make a clear distinction between material and spatial boundaries. Spatial boundaries can coincide and material boundaries cannot. They belong to different ontological categories (compare subsection 5.3.2).

6.3 Future Research

There are various directions to extend this diploma thesis. The most important future tasks are to give a model of the presented axiomatic system or subsystem (consistency) and to show that all axioms are independent.

6.3.1 Spatial Entities

One direction is the investigation of possibilities to describe or distinguish morphological properties of spatial entities. We have shown that we may distinguish a two-dimensional sphere and a torus by a formula (compare subsection 4.6.4). Is there a possibility to generalize this formula for spatial entities with an arbitrary genus? Is it possible to specify defining properties for elementary geometrical objects like a square or a triangle?

The mereotopological elementary equivalence relation provides a possibility to classify spatial entities. It is a future task to find a set of conditions $\mathcal{C}$ which is sufficient for the elementary equivalence between two spatial entities. That means two spatial entities E_1 and E_2 are mereotopological elementary equivalent if $\varphi \in \mathcal{T}(\mathcal{A}(E_1)) \Leftrightarrow \varphi \in \mathcal{T}(\mathcal{A}(E_2))$ holds for every $\varphi \in \mathcal{C}$.

We mentioned that there are analogies between graph theory and ordinary line regions. With the help of the method of interpretation one may show that line regions or a certain sub-class of line regions have a decidable or undecidable corresponding theory.

6.3.2 Material Entities

The presented theory of material entities is only a framework and no complete axiomatization. There are several unsolved questions and problems. We want to list the most important of them:

- **Material Connectedness** - We figured out that there are at least four different kinds of it. It is a future task to integrate and distinguish these kinds in our theory.
- **Granularity Shifts** - In our theory we fixed the granularity level for every single consideration. What about granularity shifts? It is future work to investigate the interrelations and variations of the corresponding sets of material parts, material boundaries, qualities and their belonging values of a certain material structure as well as the shifting of its occupied space.

- **Continuous Changes** - An important class of material entities are inhomogeneous material structures which are no compositions of homogeneous material structures. These kinds of material entities exhibit a continuous changes of a certain property value, e.g. color of a rainbow. Continuous changes have to be investigated in more detail.

- **Integration of Time** - Movements and deformations of a certain material structure as well as the distinction between presentials and persistants cannot be explained without the integration of time.

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